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INFORMATION THEORY AND INFORMATION PROCESSING

Achieving Unequal Error Protection via Woven Codes: Construction and Analysis[1]

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Abstract—In this paper several rate $R = 2/5$ convolutional encoding matrices with four states are used to illustrate how unequal error protection can be achieved for the different positions in the code sequences as well as in the information sequences. Free output- and input-distances and active burst output- and input-distances are computed. Typically a single code cannot combine a large free output- or input-distance with a steep slope. A constructive way to obtain powerful "unequal" coding schemes that provide both large distances and steep slopes is to combine several constituent encoders. Various woven schemes are studied and lower bounds on the active burst output- and input-distances are derived.

Keywords—Unequal error protection, free output-distance, free input-distance, active burst output-distance, active burst input-distance, active distance-slope, woven convolutional codes.

1. INTRODUCTION

Often in communication systems some parts of the information have to be received more reliably, that is, need better protection than others. For example, in packet transmission the header usually needs to be highly protected. A simple manner to obtain unequal error protection (UEP) is to use separate coding schemes for the parts of the information sequence that need different protection. It is, however, more challenging to design a coding scheme that provides different protection either for the different positions in the code sequences or for the different positions in the information sequences.

In [1], we studied the free output-distance and the free input-distance as the unequal counterpart to the free distance of a convolutional code. Furthermore, we also extended the active burst distance [2, 3] to their unequal-counterparts, namely, active burst output-distance and active burst input-distance. In Section 2, we compare several rate $R = 2/5$ convolutional encoding matrices and show that usually we have a trade-off between a large free output-distance and a steep active output-distance-slope. In Section 3, we study the same encoding matrices from an input-point-of-view and show that also when the least protected input has a free distance equal to the free distance of the optimum convolutional code of the same rate and complexity, we can have an even better protection of the other input. Again we see the tendency of a trade-off between a large free input-distance and a steep asymptotic slope of the active burst input-distance. We also give an example of a rate $R = 2/5$ systematic convolutional encoding matrix whose both active input-distance-slopes are equal to the best possible active distance-slope of an optimum free distance code of the same rate and complexity. Moreover, we show the somewhat surprising fact that equal error protection for code symbols does not necessarily lead to equal protection for information symbols. In order to obtain coding schemes that combine large free output-distances (free input-distances) with steep active output-distance-slopes (active

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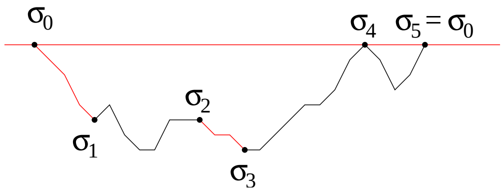

Figure 1. A path through a trellis.

input-distance-slopes) we consider various woven constructions in Section 4. Lower bounds on the active burst output- and input-distances are also derived. Some decoding issues are discussed in Section 5.

2. UNEQUAL ERROR PROTECTION OF CODE SYMBOLS

Suppose that a convolutional code $\mathcal{C}$ is encoded by an encoding matrix $G(D)$. The state-transition diagram of a realization of $G(D)$ is a directed graph $\mathcal{G}(\mathcal{S},\mathcal{U})$ with a set of vertices (states) $\mathcal{S}$ and a set of edges (transitions) $\mathcal{U}$. In the sequel we suppress, for the sake of notational simplicity, the sets of vertices and edges when they are clear from the context. Every information sequence $\boldsymbol{u}$ corresponds to a sequence of edges that form a *path* through the graph $\mathcal{G}$. Each edge is labeled by the corresponding input/output symbols. Consider a linear subcode $\mathcal{C}^{\mathrm{out}(i)}$ of a convolutional code $\mathcal{C}$ encoded by an encoding matrix $G(D)$ that consists of a restricted subset of code sequences $\boldsymbol{v}$ such that the ith output sequence is fixed to be zero, that is, $\boldsymbol{v}^{(i)} = \mathbf{0}$. These sequences correspond to the *zero paths for the ith output* in the graph $\mathcal{G}$. The subgraph $\mathcal{G}^{0,\mathrm{out}(i)} \subset \mathcal{G}$, generated by the subcode $\mathcal{C}^{\mathrm{out}(i)}$, is referred to as the *zero subgraph for the ith output.* The paths in this subgraph do not generate any weight for the ith output sequence. We call the remaining part of the graph $\mathcal{G}$, that is, the graph with edges that are the complements to those in $\mathcal{G}^{0,\mathrm{out}(i)}$ with respect to $\mathcal{G}$, the *burst subgraph for the ith output* and denote it $\mathcal{G}^{\mathrm{b},\mathrm{out}(i)}$. These two subgraphs do not have any edges in common. All vertices in the zero subgraph are also vertices in the burst subgraph but not vice versa. However, all states in the graph $\mathcal{G}$ are also states in burst subgraph $\mathcal{G}^{\mathrm{b},\mathrm{out}(i)}$ for all outputs i.

Consider all pairs of distinct vertices σ_k and σ_l from the zero subgraph for the ith output. For all such pairs of vertices, we construct a set $\mathcal{P}^{0,\mathrm{out}(i)}$ which contains all path segments through the zero subgraph for the ith output $\mathcal{G}^{0,\mathrm{out}(i)}$, starting at σ_k, terminating at σ_l, and having the smallest Hamming weight of the corresponding segment of all code sequences $\boldsymbol{v}$. We also introduce the set $\mathcal{P}^{\mathrm{b},\mathrm{out}(i)}$ of all paths through the burst subgraph for the ith output $\mathcal{G}^{\mathrm{b},\mathrm{out}(i)}$, starting at σ_k and terminating at σ_l. Consider a path with a finite span [3] that does not contain segments of the all-zero path within the span. If such a path contains one or more path segments from the set $\mathcal{P}^{\mathrm{b},\mathrm{out}(i)}$, and if the path segments between them and possibly the diverging and remerging segments all are from the $\mathcal{P}^{0,\mathrm{out}(i)}$, then we this path a *detour for the ith output.* We denote the set of all such detours for the ith output by $\mathcal{D}^{\mathrm{out}(i)}$.

A path through the trellis induced by the graph $\mathcal{G}$ is shown in Figure 1. The segments drawn by red lines correspond to path segments from the zero subgraph for the ith output, $\mathcal{G}^{0,\mathrm{out}(i)}$, and the black lines correspond to path segments from the burst subgraph $\mathcal{G}^{\mathrm{b},\mathrm{out}(i)}$. If the path segments $\sigma_0 \to \sigma_1$ and $\sigma_2 \to \sigma_3$ both are from $\mathcal{P}^{0,\mathrm{out}(i)}$, then this path with path segments $\sigma_1 \to \sigma_2$, $\sigma_3 \to \sigma_4$, and $\sigma_4 \to \sigma_5 = \sigma_0$ from $\mathcal{G}^{\mathrm{b},\mathrm{out}(i)}$ is a detour for the ith output in $\mathcal{D}^{\mathrm{out}(i)}$. Note that this detour does not end at the vertex σ_4, but at σ_5; only from that moment on it stays remerged with the all-zero path.

Let $\mathcal{D}_l^{\mathrm{out}(i)}$ be the set of detours for the ith output of length l, then the set $\mathcal{D}^{\mathrm{out}(i)}$ is the union of the detour sets $\mathcal{D}_l^{\mathrm{out}(i)}$, that is,

$$\mathcal{D}^{\mathrm{out}(i)} = \bigcup_l \mathcal{D}_l^{\mathrm{out}(i)}. \tag{1}$$

Using the set of detours $\mathcal{D}_{j+1}^{\mathrm{out}(i)}$ we extended the definition of the active burst distance such that it will characterize the ith output sequences for a given convolutional code [1].

Definition 1. For a rate $R = b/c$ convolutional code $\mathcal{C}$ the jth order active burst distance for the ith output is

$$a_j^{\mathrm{b,out}(i)} = \min_{\boldsymbol{v} \in \mathcal{D}_{j+1}^{\mathrm{out}(i)}} \Big\{ w_{\mathrm{H}}(\boldsymbol{v}) \Big\}. \tag{2}$$

The minimum normalized weight of a cycle, that contains at least one segment from $\mathcal{P}^{\mathrm{b,out}(i)}$ and only path segments from $\mathcal{P}^{0,\mathrm{out}(i)}$ are used to connect the segments from $\mathcal{P}^{\mathrm{b,out}(i)}$, is called the *active distance-slope for the ith output* $\alpha^{\mathrm{out}(i)}$. The active burst distance for the ith output is lower-bounded by an affine function with slope $\alpha^{\mathrm{out}(i)}$. Moreover, the minimum weight of a detour from the all-zero path with a nonzero ith output is the free distance for the ith output [1]. The active burst distance for the ith output is lower-bounded by

$$a_j^{\mathrm{b,out}(i)} \geq \breve{a}_j^{\mathrm{b,out}(i)} \triangleq \max \Big\{ d_{\mathrm{free}}^{\mathrm{out}(i)}, \alpha^{\mathrm{out}(i)} j + \beta^{\mathrm{b,out}(i)} \Big\}, \tag{3}$$

where $\beta^{\mathrm{b,out}(i)}$ is chosen as large as possible and $d_{\mathrm{free}}^{\mathrm{out}(i)}$ is the *free distance for the ith output,* which was defined in [1] as

$$d_{\mathrm{free}}^{\mathrm{out}(i)} = \min_{\boldsymbol{v}^{(i)} \neq \mathbf{0}} \Big\{ w_{\mathrm{H}}(\boldsymbol{v}) \Big\}. \tag{4}$$

The active output-distances do not depend on the mapping between the information sequences $\boldsymbol{u}$ and the corresponding code sequences $\boldsymbol{v}$. They are code properties.

Example 1. Consider the rate $R = 2/5$ convolutional codes encoded by the following minimal-basic encoding matrices of memory $m = 1$

$$G_1(D) = \begin{pmatrix} 1+D & 0 & 1+D & 1 & D \\ 0 & 1+D & D & 1+D & 1+D \end{pmatrix}, \tag{5}$$

$$G_2(D) = \begin{pmatrix} D & D & 0 & 1+D & 1 \\ 1 & 1+D & 1+D & 1 & 1+D \end{pmatrix}, \tag{6}$$

$$G_3(D) = \begin{pmatrix} 1 & 0 & 0 & D & 1+D \\ 1+D & 1+D & 1+D & 1+D & 1 \end{pmatrix}, \tag{7}$$

and

$$G_4(D) = \begin{pmatrix} 0 & 0 & 0 & 1 & 1+D \\ 1+D & 1+D & 1+D & 1+D & D \end{pmatrix}. \tag{8}$$

Their active output-distances are shown in Figures 2–5. In Table 1, we give the corresponding values of the free output-distances, $d_{\mathrm{free}}^{\mathrm{out}(i)}$, and the active output-distance-slopes, $\alpha^{\mathrm{out}(i)}$.

From these figures one can easily see that the idea of comparing two encoding matrices or the outputs of a single encoding matrix only by using the free output-distances is not really useful. In addition to the values of $d_{\mathrm{free}}^{\mathrm{out}(i)}$, we have to take into account the corresponding active output-distances, which makes our analysis more adequate.

Example 2. Consider the rate $R = 2/5$ convolutional codes encoded by the following polynomial, systematic encoding matrices with overall constraint length $\nu = 2$

$$G_{\mathrm{sys},1}(D) = \begin{pmatrix} 1 & 0 & 1+D & 0 & D \\ 0 & 1 & D & 1+D & 1+D \end{pmatrix} \tag{9}$$

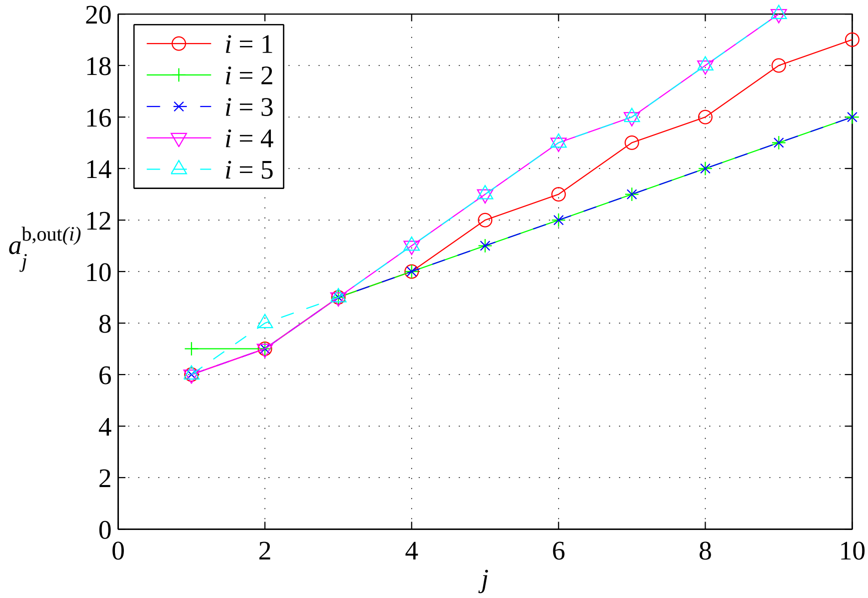

Figure 2. Active burst output-distances for $G_1(D)$.

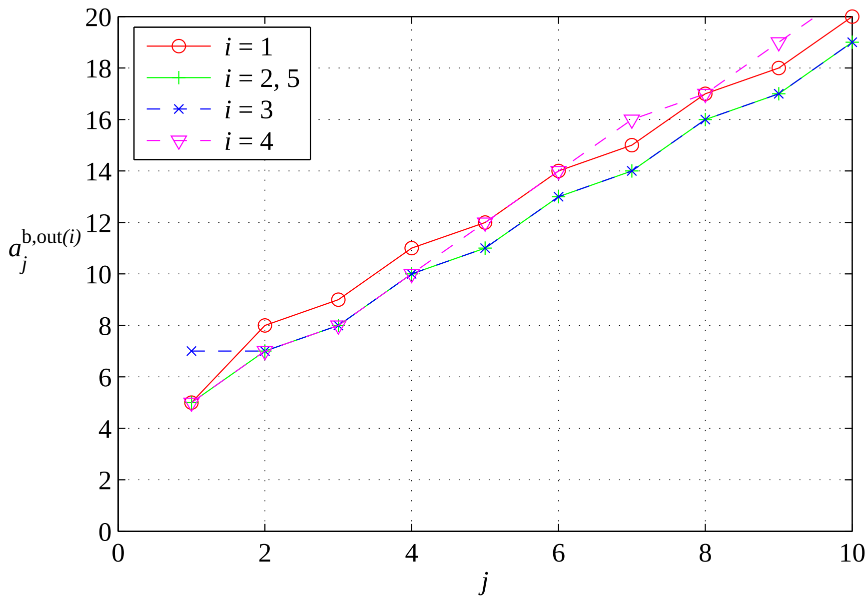

Figure 3. Active burst output-distances for $G_2(D)$.

and

$$G_{\mathrm{sys},2}(D) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1+D+D^2 & 1+D^2 & D+D^2 \end{pmatrix}. \tag{10}$$

Their active input-distances are shown in Figures 6 and 7, respectively, and the values of the free input-distances as well as the active input-distance-slopes are presented in Table 2.

The encoding matrix $G_{\mathrm{sys},1}(D)$ is an interesting case since all outputs have the same active output-distance-slope $\alpha^{\mathrm{out}(i)} = 2, i = 1, \ldots, 5$, as the maximum possible active distance-slope $\alpha = 2$ of an

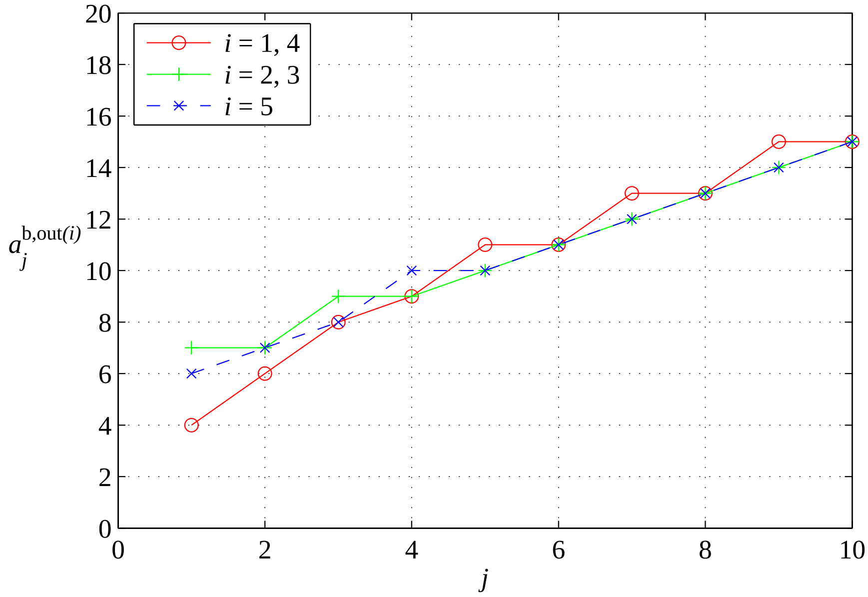

Figure 4. Active burst output-distances for $G_3(D)$.

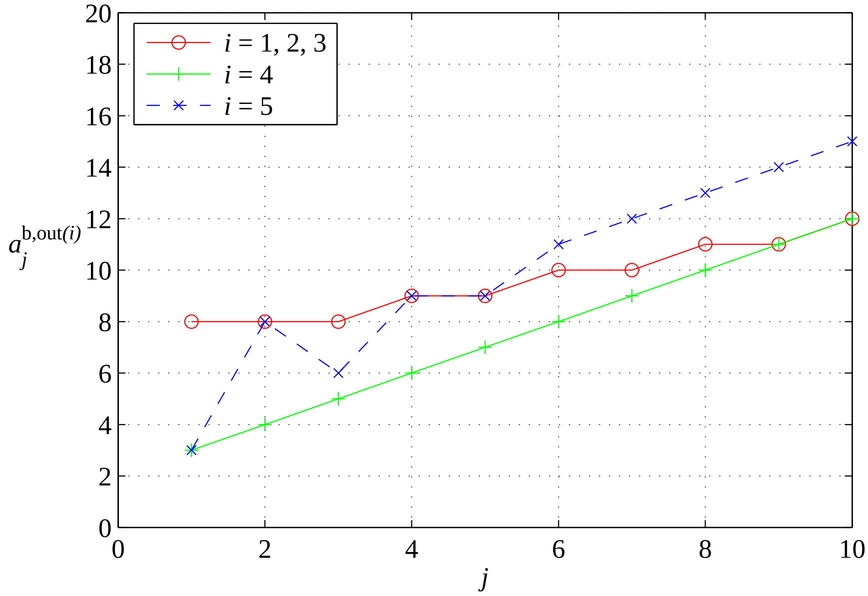

Figure 5. Active burst output-distances for $G_4(D)$.

optimum free distance (OFD) code with systematic, polynomial encoding matrix

$$G_{\mathrm{sys,OFD}}(D) = \begin{pmatrix} 1 & 0 & D & 1 & 1+D \\ 0 & 1 & 1+D & 1+D & D \end{pmatrix}. \tag{11}$$

At the same time, the free output-distance of $G_{\mathrm{sys},1}(D)$ is $\boldsymbol{d}_{\mathrm{free}}^{\mathrm{out}} = (4, 6, 4, 6, 4)$, while $G_{\mathrm{sys,OFD}}(D)$ has $d_{\mathrm{free}} = 5$. Thus, we are able to "rearrange" the free output-distance without damaging the active distance-slope. Unfortunately, this is rather an exception than a common feature and in general encoding matrices with unequal error protection have smaller active distance-slope than the maximum possible α for a corresponding OFD code.

	i	1	2	3	4	5
$G_1(D)$	$d_{\mathrm{free}}^{\mathrm{out}(i)}$	6	7	6	6	6
	$\alpha^{\mathrm{out}(i)}$	3/2	1	1	2	2
$G_2(D)$	$d_{\mathrm{free}}^{\mathrm{out}(i)}$	5	5	7	5	5
	$\alpha^{\mathrm{out}(i)}$	3/2	3/2	3/2	2	3/2
$G_3(D)$	$d_{\mathrm{free}}^{\mathrm{out}(i)}$	4	7	7	4	4
	$\alpha^{\mathrm{out}(i)}$	1	1	1	1	1
$G_4(D)$	$d_{\mathrm{free}}^{\mathrm{out}(i)}$	8	8	8	3	3
	$\alpha^{\mathrm{out}(i)}$	1/2	1/2	1/2	1/2	1

Table 1. Parameters for the lower bounds on active output-distances for the encoding matrices given in Example 1.

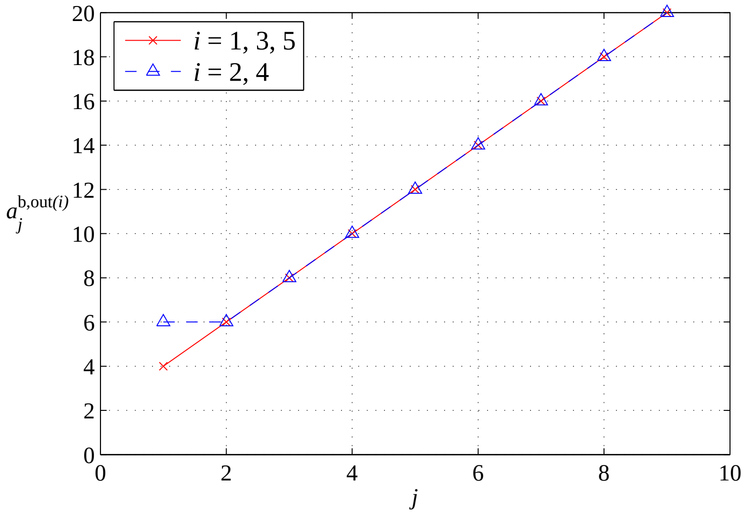

Figure 6. Active burst output-distances for $G_{\mathrm{sys},1}(D)$.

3. UNEQUAL ERROR PROTECTION OF INFORMATION SYMBOLS

Consider all information sequences $\boldsymbol{u}$ such that the ith input sequence is all-zero, that is, $\boldsymbol{u}^{(i)} = \boldsymbol{0}$. These sequences determine the *zero paths for the ith input* in the graph $\mathcal{G}$. Let $\mathcal{S}^{\mathrm{in}(i)}$ denote the set of vertices that are strongly connected with the all-zero vertex by zero paths for the ith input. Let $\mathcal{U}^{\mathrm{in}(i)}$ be the set of edges from those zero paths for the ith input that connect the vertices in $\mathcal{S}^{\mathrm{in}(i)}$. The information sequences $\boldsymbol{u}$ with $\boldsymbol{u}^{(i)} = \boldsymbol{0}$ generate a subgraph $\mathcal{G}^{0,\mathrm{in}(i)}$, which we call the *zero subgraph for the ith input.* We call the subgraph with edges that are the complement to those in $\mathcal{G}^{0,\mathrm{in}(i)}$ with respect to $\mathcal{G}$, the *burst subgraph for the ith input* and denote it $\mathcal{G}^{\mathrm{b},\mathrm{in}(i)}$.

Note that when considering state-transition graphs with respect to the inputs of the corresponding encoding matrix, the mapping between the information sequences $\boldsymbol{u}$ and code sequences $\boldsymbol{v}$ becomes important. Hence, to obtain the desired distance properties for the inputs one has to carefully choose the encoding matrix, not only the code. But for the distance properties for outputs only the choice of the convolutional code is crucial.

Consider all pairs of distinct vertices σ_k and σ_l from the zero subgraph for the ith input, that is, $\sigma_k, \sigma_l \in \mathcal{S}^{\mathrm{in}(i)}$, $\sigma_k \neq \sigma_l$. For all such pairs of vertices σ_k and σ_l, we construct a set $\mathcal{P}^{0,\mathrm{in}(i)}$ which contains all path segments through the zero subgraph for the ith input $\mathcal{G}^{0,\mathrm{in}(i)}$, starting at σ_k, terminating at σ_l, and having the smallest Hamming weight of the corresponding segment of all code sequences $\boldsymbol{v}$. We also introduce the set $\mathcal{P}^{\mathrm{b},\mathrm{in}(i)}$ of all paths through the burst subgraph for the ith input $\mathcal{G}^{\mathrm{b},\mathrm{in}(i)}$, starting at some vertex $\sigma_k \in \mathcal{S}^{\mathrm{in}(i)}$ and terminating at some vertex $\sigma_l \in \mathcal{S}^{\mathrm{in}(i)}$.

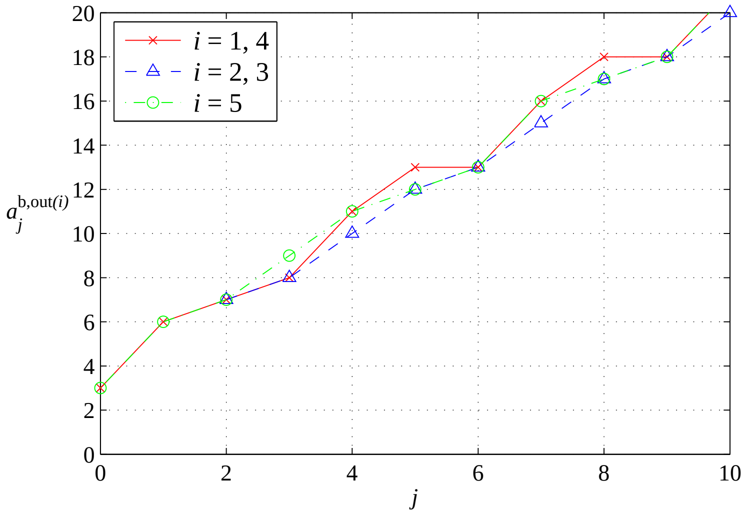

Figure 7. Active burst output-distances for $G_{sys,2}(D)$.

	i	1	2	3	4	5
$G_{sys,1}(D)$	$d_{\rm free}^{{\rm out}(i)}$	4	6	4	6	4
	$\alpha^{{\rm out}(i)}$	2	2	2	2	2
$G_{sys,2}(D)$	$d_{\rm free}^{{\rm out}(i)}$	3	7	7	3	3
	$\alpha^{{\rm out}(i)}$	5/3	5/3	5/3	5/3	5/3

Table 2. Parameters for the lower bounds on active output-distances for the encoding matrices given in Example 2.

Consider a path with a finite span that does not contain segments of the all-zero path within the span. If this path contains one or more path segments from the set $\mathcal{P}^{{\rm b,in}(i)}$, and if the path segments between them and possibly the diverging and remerging segments all are from the $\mathcal{P}^{0,{\rm in}(i)}$, then we call such a path a *detour for the ith input.* We denote the set of all such detours for the ith input, $\mathcal{D}^{{\rm in}(i)}$.

Let $\mathcal{D}_l^{{\rm in}(i)}$ be the set of detours for the ith input of length l, then the set $\mathcal{D}^{{\rm in}(i)}$ is the union of the detour sets $\mathcal{D}_l^{{\rm in}(i)}$, that is,

$$\mathcal{D}^{{\rm in}(i)} = \bigcup_l \mathcal{D}_l^{{\rm in}(i)}. \tag{12}$$

Using the set of detours $\mathcal{D}_{j+1}^{{\rm in}(i)}$ we extended in [1] the definition of the active burst distance [2, 3] such that it will characterize the ith input sequences for a given encoding matrix.

Definition 2. For a rate $R = b/c$ convolutional code $\mathcal{C}$ with encoding matrix $G(D)$ the jth order active burst distance for the ith input is

$$a_j^{{\rm b,in}(i)} = \min_{\boldsymbol{v}\in\mathcal{D}_{j+1}^{{\rm in}(i)}} \left\{ w_{\rm H}(\boldsymbol{v}) \right\}. \tag{13}$$

The active burst distance for the ith input is lower-bounded by an affine function with slope $\alpha^{{\rm in}(i)}$, the *active distance-slope for the ith input.* It can be determined as the minimum normalized weight for a cycle that contains at least one segment from $\mathcal{P}^{{\rm b,in}(i)}$ and only path segments from $\mathcal{P}^{0,{\rm in}(i)}$ are used to connect the segments from $\mathcal{P}^{{\rm b,in}(i)}$. For any noncatastrophic encoding matrix $G(D)$, the minimum weight of a detour from the all-zero path caused by a nonzero ith input is equal to the free distance for the ith input [4, 5, 1],

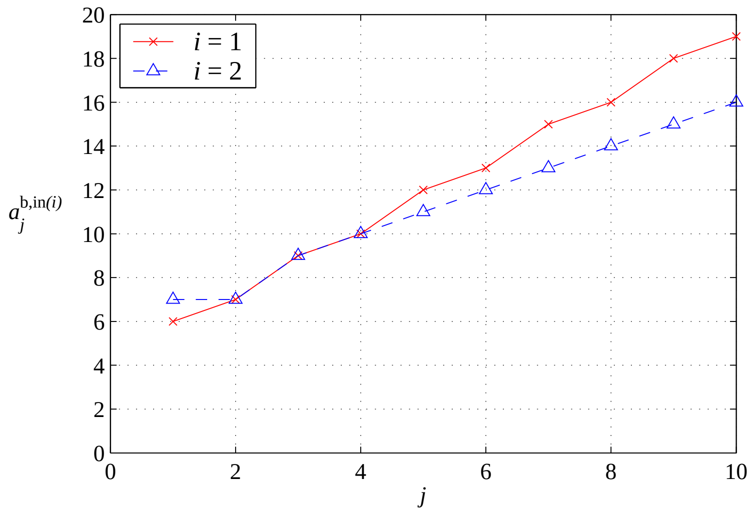

Figure 8. Active burst input-distances for $G_1(D)$.

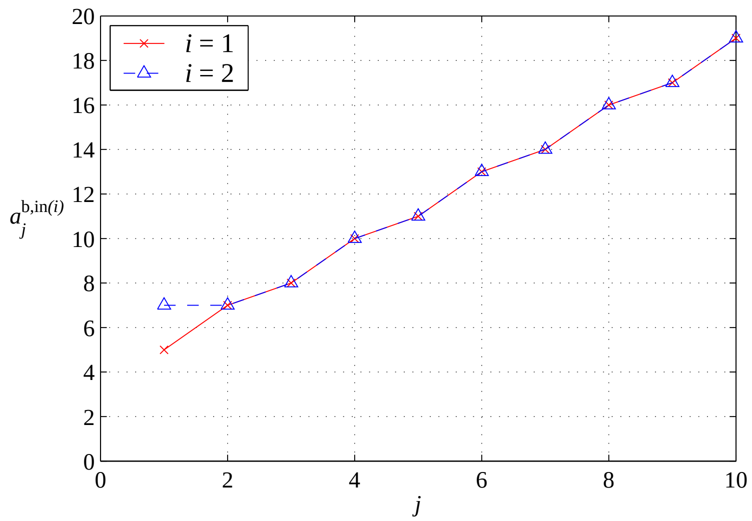

Figure 9. Active burst input-distances for $G_2(D)$.

and we have the following lower bound on the active burst distance for the ith input

$$a_j^{\mathrm{b,in}(i)} \geq \breve{a}_j^{\mathrm{b,in}(i)} \triangleq \max\left\{d_{\mathrm{free}}^{\mathrm{in}(i)}, \alpha^{\mathrm{in}(i)} j + \beta^{\mathrm{b,in}(i)}\right\}, \tag{14}$$

where $\beta^{\mathrm{b,in}(i)}$ is chosen as large as possible.

Unlike the active output-distances, the active input-distances do depend on the actual mapping between the information sequences and the code sequences. Hence, they are encoding matrix properties.

Example 3. Consider again the rate $R = 2/5$ convolutional encoding matrices given in Example 1. Their active input-distances are shown in Figures 8–11 and the values of the free input-distances, active input-distance-slopes are presented in Table 3.

Note that $G_1(D)$ outperforms the following encoding matrix given in [6]

$$G_{\mathrm{Mills}}(D) = \begin{pmatrix} D & 1 & D & 1+D & 1 \\ 1+D & 1+D & 1+D & 0 & D \end{pmatrix}. \tag{15}$$

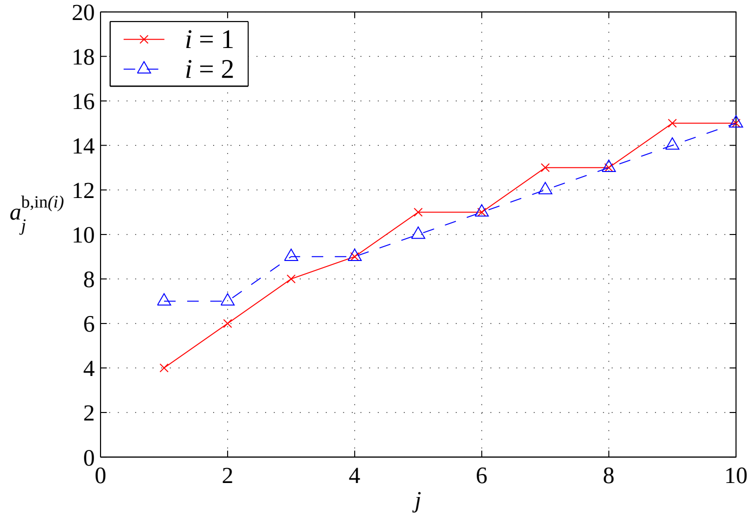

Figure 10. Active burst input-distances for $G_3(D)$.

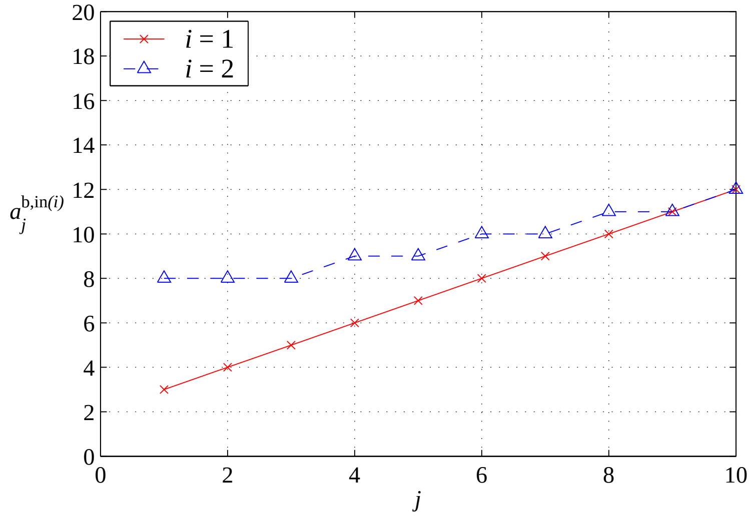

Figure 11. Active burst input-distances for $G_4(D)$.

Both matrices have the same free input-distance $\boldsymbol{d}_{\mathrm{free}}^{\mathrm{in}} = (6, 7)$, but $G_1(D)$ has active input-distance-slopes $\alpha^{\mathrm{in}(1)} = 3/2$ and $\alpha^{\mathrm{in}(2)} = 1$, while $G_{\mathrm{Mills}}(D)$ has only $\alpha^{\mathrm{in}(1)} = \alpha^{\mathrm{in}(2)} = 1/2$.

Figure 8 easily demonstrates that it is sometimes difficult to specify which input is better protected. On one hand, the second input has higher free input-distance, which guarantees better performance at high signal-to-noise ratios. On the other hand, the active distance-slope for the second input is less than that for the first input, thus, for rather bad channels when the error bursts tend to be longer the second input performs worse then the first one.

It is interesting that for all these encoding matrices the $d_{\mathrm{free}}^{\mathrm{in}(2)}$ is larger than the upper bound on the free distance obtained from the Heller bound [7, 8, 3].

A slightly more general form of the Heller bound on the free distance for any binary, rate $R = b/c$ convolutional code $\mathcal{C}$ encoded by a minimal-basic encoding matrix of memory m and overall constraint

	i	1	2
$G_1(D)$	$d_{\text{free}}^{\text{in}(i)}$	6	7
	$\alpha^{\text{in}(i)}$	3/2	1
$G_2(D)$	$d_{\text{free}}^{\text{in}(i)}$	5	7
	$\alpha^{\text{in}(i)}$	3/2	3/2
$G_3(D)$	$d_{\text{free}}^{\text{in}(i)}$	4	7
	$\alpha^{\text{in}(i)}$	1	1
$G_4(D)$	$d_{\text{free}}^{\text{in}(i)}$	3	8
	$\alpha^{\text{in}(i)}$	1	1/2

Table 3. Parameters for the lower bounds on active input-distances for the encoding matrices given in Example 1.

length ν is given in [3]:

$$d_{\text{free}} \leq \min_{i\geq 1} \left\{ \left\lfloor \frac{(m+i)c}{2\left(1-2^{\nu-b(m+i)}\right)} \right\rfloor \right\}. \tag{16}$$

Substituting the corresponding parameters, we obtain that for any $R = 2/5$ convolutional code $\mathcal{C}$ encoded by a minimal-basic encoding matrix with overall constraint length $\nu = 2$ the free distance is upper-bounded by

$$d_{\text{free}} \leq 6, \tag{17}$$

when $\nu_1 = \nu_2 = 1$, and

$$d_{\text{free}} \leq 8, \tag{18}$$

when $\nu_1 = 0, \nu_2 = 2$.

Investigating unequal error protection for information and code symbols, immediately leads to the following question: Which free input- and output-distances are possible for a rate $R = b/c$ convolutional code $\mathcal{C}$ encoded by a minimal-basic encoding matrix $G(D)$ of overall constraint length ν.

Unequal error protection for information symbols of convolutional codes was studied by Mills [6]. In particular, she constructed upper bounds on the free input-distances similar to the Plotkin bound and the Griesmer bound for block codes. Consider a rate $R = b/c$ convolutional code $\mathcal{C}$ encoded by a minimal-basic encoding matrix $G(D)$ with memory m. If $d_{\text{free}}^{\text{in}(i)}$ and $d_{\text{free}}^{\text{in}(j)}$ are the free distances for the ith and the jth input, respectively, then we have Mills' upper bound

$$d_{\text{free}}^{\text{in}(j)} \leq \left\lfloor \frac{2c(m+1) - d_{\text{free}}^{\text{in}(i)}}{2} \right\rfloor. \tag{19}$$

According to this bound, the free input-distance for a rate $R = 2/5$ convolutional code encoded by a minimal-basic encoding matrix with constraint lengths $\nu_1 = \nu_2 = 1$ can be at most $\boldsymbol{d}_{\text{free}}^{\text{in}} = (6,7)$, $\boldsymbol{d}_{\text{free}}^{\text{in}} = (5,7)$, $\boldsymbol{d}_{\text{free}}^{\text{in}} = (4,8)$, or $\boldsymbol{d}_{\text{free}}^{\text{in}} = (3,8)$. An exhaustive search over all minimal-basic encoding matrices of memory $m = 1$ proves that this bound is tight and all points except $\boldsymbol{d}_{\text{free}}^{\text{in}} = (4,8)$ are indeed reachable as shown in the previous example.

Consider a special case of polynomial, systematic encoding matrices. The free distance for any binary, rate $R = b/c$ convolutional code encoded by a polynomial, systematic encoding matrix of memory m satisfies [3]

$$d_{\text{free}} \leq \min_{i\geq 1} \left\{ \left\lfloor \frac{(m(1-R)+i)c}{2\left(1-2^{-bi}\right)} \right\rfloor \right\}. \tag{20}$$

Thus, for a rate $R = 2/5$ convolutional code with a polynomial, systematic encoding matrix of overall constraint length $\nu = 2$ the free distance is upper-bounded by $d_{\text{free}} \leq 5$.

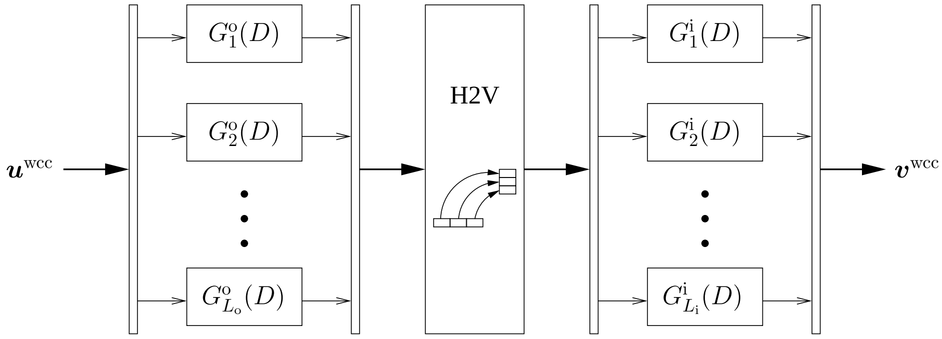

Figure 12. A woven convolutional encoder.

By exhaustive search among all polynomial, systematic encoding matrices of memory $m = 1$ we find that there exists no encoding matrix with $d_{\mathrm{free}}^{\mathrm{in}(1)} = 5$ and $d_{\mathrm{free}}^{\mathrm{in}(2)} \geq 5$. If we fix $d_{\mathrm{free}}^{\mathrm{in}(1)} = 4$, then, as shown by the following example, there exist indeed some encoding matrices providing unequal error protection for information symbols.

Example 4. Consider again the encoding matrices given in Example 2. Since these encoding matrices are systematic, the zero subgraphs for the inputs coincide with zero subgraphs for the first two (the "systematic") outputs. Thus, the active input-distances coincide with the first two active output-distances. We refer to Figures 6 and 7 for the active input-distances and to Table 2 for the free input-distances as well as the active input-distance-slopes.

An important difference between the active input-distance and the active output-distance is that the former is the property of a given encoding matrix $G(D)$, while the latter is a property of the corresponding convolutional code $\mathcal{C}$ and for a given code does not depend on the choice of the actual encoding matrix.

4. WOVEN SCHEMES FOR UNEQUAL ERROR PROTECTION

Similarly to the situation for free distances and active-distance slopes, convolutional encoding matrices with large free input-distances (free output-distances) typically have low active input-distance-slopes (active output-distance-slopes) or do not provide any UEP at all. It is, however, rather simple to construct encoding schemes that provide unequal error protection.

In [9], unequal error protection was achieved by a scheme with parallel concatenation of different constituent convolutional encoders. In [10], Jordan *et al.* proposed the use of a woven convolutional encoder [11] for unequal error protection for both information and code symbols. Next we take a closer look at woven convolutional encoders and derive theoretical lower bounds on their input- and output-error-correcting capabilities.

4.1. Woven Convolutional Encoders

Woven convolutional codes were originally presented and studied in [11, 12]. A woven convolutional encoder is a serial concatenation of an inner warp and an outer warp. A warp is a bank of parallel constituent encoders. Such a scheme, sometimes called a twill, is shown in Figure 12.

In a woven encoding scheme the information sequence $\boldsymbol{u}^{\mathrm{wcc}}$ is subdivided according to the number of inputs of the constituent encoders in the outer warp and independently encoded by these encoders. The resulting sequence of every constituent encoder is serialized and written row-wise into a buffer which consists of L_{o} rows. Then, the data is read from the buffer column-wise and fed to the constituent encoders in the inner warp. This horizontal-to-vertical permutation is denoted by H2V in Figure 12. The constituent encoders in the inner warp operate independently and the resulting code sequences are serialized and multiplexed into a resulting sequence $\boldsymbol{v}^{\mathrm{wcc}}$.

If the outer warp of a twill consists of L_{o} identical encoders with (semi-infinite) encoding matrices G^{o}, and the inner warp consists of L_{i} identical encoders with (semi-infinite) encoding matrices G^{i}, then such a twill has the following (semi-infinite) encoding matrix

$$G^{\mathrm{tw}} = (G^{\mathrm{o}} \otimes I_{L_{\mathrm{o}}}) \left(G^{\mathrm{i}} \otimes I_{L_{\mathrm{i}}}\right) \tag{21}$$

where $I_{L_{\mathrm{o}}}$ and $I_{L_{\mathrm{i}}}$ are the $L_{\mathrm{o}} \times L_{\mathrm{o}}$ and the $L_{\mathrm{i}} \times L_{\mathrm{i}}$ identity matrices, respectively, $\otimes$ denotes the matrix direct product[1].

If the number of outputs of the outer warp c^{ow} matches to the number of inputs in the inner warp b^{iw}, then the encoding matrix of the twill in D-domain, $G^{\mathrm{tw}}(D)$, can be written in a similar form. If the warps do not match, an additional blocking [14] of the constituent encoders can be performed that enlarges both the number of inputs and outputs, but keeps the binary mapping between them the same. The required blocking factors are

$$B^{\mathrm{ow}} = \frac{1}{c^{\mathrm{ow}}} \operatorname{lcm}\left(c^{\mathrm{ow}}, b^{\mathrm{iw}}\right) \quad \text{and} \quad B^{\mathrm{iw}} = \frac{1}{b^{\mathrm{iw}}} \operatorname{lcm}\left(c^{\mathrm{ow}}, b^{\mathrm{iw}}\right). \tag{22}$$

Then, after appropriate blocking, the encoding matrix for a twill with identical constituent encoders in the warps is given by

$$G^{\mathrm{tw}}(D) = \left(G^{\mathrm{o}}(D) \otimes I_{L_{\mathrm{o}}}\right)^{[B^{\mathrm{ow}}]} \left(G^{\mathrm{i}}(D) \otimes I_{L_{\mathrm{i}}}\right)^{[B^{\mathrm{iw}}]} \tag{23}$$

If the number of constituent encoders L_{o} and L_{i} are relatively prime the twill works as a single entity and does not split into disjoint parts. The two degenerated cases when the outer warp and the inner warp consist of a single constituent encoder are called woven convolutional encoders with inner and with outer warp, respectively.

Example 5. Consider a twill with an outer warp consisting of two encoders with encoding matrix

$$G^{\mathrm{o}}(D) = \begin{pmatrix} 1 & 1 + D \end{pmatrix} \tag{24}$$

and an inner warp consisting of two encoders with encoding matrix

$$G^{\mathrm{i}}(D) = \begin{pmatrix} 1 + D^2 & 1 + D + D^2 \end{pmatrix}. \tag{25}$$

The blocking factors are

$$B^{\mathrm{ow}} = \frac{1}{L_{\mathrm{o}} c^{\mathrm{o}}} \operatorname{lcm}\left(L_{\mathrm{o}} c^{\mathrm{o}}, L_{\mathrm{i}} b^{\mathrm{i}}\right) = \frac{1}{4} \operatorname{lcm}(4, 2) = 1$$

and

$$B^{\mathrm{in}} = \frac{1}{L_{\mathrm{i}} b^{\mathrm{i}}} \operatorname{lcm}\left(L_{\mathrm{o}} c^{\mathrm{o}}, L_{\mathrm{i}} b^{\mathrm{i}}\right) = \frac{1}{2} \operatorname{lcm}(4, 2) = 2.$$

---

[1] The matrix direct (Kronecker) product [13] of two matrices A and B is given by $A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{k1}B & \cdots & a_{kn}B \end{pmatrix}$.

Then, according to (23), the encoding matrix for the woven encoder is

$$G^{\mathrm{tw}}(D) = \left(G^{\mathrm{o}}(D) \otimes I_2\right)\left(G^{\mathrm{i}}(D) \otimes I_2\right)^{[2]}$$

$$= \begin{pmatrix} 1 & 0 & 1+D & 0 \\ 0 & 1 & 0 & 1+D \end{pmatrix} \begin{pmatrix} 1+D & 0 & 1+D & 0 & 0 & 0 & 1 & 0 \\ 0 & 1+D & 0 & 1+D & 0 & 0 & 0 & 1 \\ 0 & 0 & D & 0 & 1+D & 0 & 1+D & 0 \\ 0 & 0 & 0 & D & 0 & 1+D & 0 & 1+D \end{pmatrix}$$

$$= \begin{pmatrix} 1+D & 1+D^2 & 0 & 0 & 1+D^2 & D^2 & 0 & 0 \\ 0 & 0 & 1+D & 1+D^2 & 0 & 0 & 1+D^2 & D^2 \end{pmatrix}.$$

Even such a simple encoding scheme has many outputs. However, since L_{o} and L_{i} are not relatively prime, this scheme behaves as two parallel encoders: one with outputs $1, 2, 5$, and 6 and another one with outputs $3, 4, 7$, and 8.

In the case of different constituent encoders, (23) has to be rewritten as

$$G^{\mathrm{tw}}(D) = \left(\sum_{l=1}^{L_{\mathrm{o}}} G_l^{\mathrm{o}}(D) \otimes \mathrm{diag}(\boldsymbol{e}_l)\right)^{[B^{\mathrm{ow}}]} \left(\sum_{l=1}^{L_{\mathrm{i}}} G_l^{\mathrm{i}}(D) \otimes \mathrm{diag}(\boldsymbol{e}_l)\right)^{[B^{\mathrm{iw}}]} \tag{26}$$

where $\mathrm{diag}(\boldsymbol{e}_l)$ is a diagonal matrix with the lth unit vector on the main diagonal.

Using the notation introduced in [11], $j_{2\mathrm{free}}^{\mathrm{b}}$ denotes the smallest j for which the lower bound on the active burst distance $\breve{a}_j^{\mathrm{b}}$ is at least twice the free distance, that is, the smallest j such that

$$\breve{a}_j^{\mathrm{b}} \geq 2d_{\mathrm{free}}. \tag{27}$$

Let a_j^{s} denote the active segment distance [2,3]. We write the corresponding lower bound as $\breve{a}_j^{\mathrm{s}}$. Analogously to $j_{2\mathrm{free}}^{\mathrm{b}}$, let $j_{\mathrm{free}}^{\mathrm{s}}$ denote the smallest j for which

$$\breve{a}_j^{\mathrm{s}} \geq d_{\mathrm{free}}. \tag{28}$$

To distinguish between the active distances for the constituent encoders in the warps of a given woven convolutional encoder we write superscripts o and i denoting the outer warp and the inner warp, respectively, and use a subscript to denote the encoder number. For example, $a_{j,3}^{\mathrm{c,o}}$ should be read as the jth order active column distance for the third constituent encoder in the outer warp. The structure of a woven encoder allows us to lower-bound the active input- and output-distances, extending the results in [11] to the case of warps with different constituent encoders.

Theorem 1. *Consider a woven convolutional encoder with an outer warp consisting of $L_{\mathrm{o},1}$ encoders for rate $R_1^{\mathrm{o}} = b_1^{\mathrm{o}}/c_1^{\mathrm{o}}$ encoding matrices $G_1^{\mathrm{o}}(D)$ and $L_{\mathrm{o},2}$ encoders for rate $R_2^{\mathrm{o}} = b_2^{\mathrm{o}}/c_2^{\mathrm{o}}$ encoding matrices $G_2^{\mathrm{o}}(D)$. Suppose that the inner encoder corresponds to a rate $R^{\mathrm{i}} = b^{\mathrm{i}}/c^{\mathrm{i}}$ generator matrix $G^{\mathrm{i}}(D)$. Assume that the total number of constituent encoders $L_{\mathrm{o}} = L_{\mathrm{o},1} + L_{\mathrm{o},2}$ satisfies*

$$L_{\mathrm{o}} \geq \left(j_{2\mathrm{free}}^{\mathrm{b,i}} + 1\right) b^{\mathrm{i}}. \tag{29}$$

If the following restriction is fulfilled

$$\min\left(\beta_1^{\mathrm{c,o}}, \beta_1^{\mathrm{rc,o}}, \beta_2^{\mathrm{c,o}}, \beta_2^{\mathrm{rc,o}}\right) \geq \alpha^{\mathrm{o}} + 1, \tag{30}$$

then the active burst distance for the first input is lower-bounded by

$$a_j^{\mathrm{b,in}(1)} \geq d_{\mathrm{free}}^{\mathrm{i}} \alpha^{\mathrm{o}} j + d_{\mathrm{free}}^{\mathrm{i}} \beta_1^{\mathrm{b,o}}, \tag{31}$$

where α^{o} is the minimum of the two active distance-slopes for the constituent outer encoders and $d_{\mathrm{free}}^{\mathrm{i}}$ is the free distance of the inner encoder.

Proof. See Appendix.

Although only the first input of a woven encoder with outer warp is considered, the active burst distances for the other inputs can be lower-bounded and outer warps with larger variety of the constituent encoders can be considered in the same manner with an appropriate change of indices. A theorem similar to Theorem 1 can also be stated for a woven convolutional encoder with inner warp.

Theorem 2. *Consider a woven convolutional encoder with an outer encoder with a rate* $R^{\mathrm{o}} = b^{\mathrm{o}}/c^{\mathrm{o}}$ *encoding matrix* $G^{\mathrm{o}}(D)$ *and an inner warp consisting of* $L_{\mathrm{i},1}$ *encoders for rate* $R_1^{\mathrm{i}} = b_1^{\mathrm{i}}/c_1^{\mathrm{i}}$ *encoding matrices* $G_1^{\mathrm{i}}(D)$ *and* $L_{\mathrm{i},2}$ *encoders for rate* $R_2^{\mathrm{i}} = b_2^{\mathrm{i}}/c_2^{\mathrm{i}}$ *encoding matrices* $G_2^{\mathrm{i}}(D)$. *Assume that the total number of constituent encoders* $L_{\mathrm{i}} = L_{\mathrm{i},1} + L_{\mathrm{i},2}$ *satisfies*

$$L_{\mathrm{i}} \geq \left(j_{\mathrm{free}}^{\mathrm{s,o}} + 1\right) c^{\mathrm{o}}. \tag{32}$$

If the following restriction is fulfilled

$$\min\left(\beta_1^{\mathrm{c,i}}, \beta_1^{\mathrm{rc,i}}, \beta_2^{\mathrm{c,i}}, \beta_2^{\mathrm{rc,i}}\right) \geq \alpha^{\mathrm{i}}, \tag{33}$$

then the active burst distance for the first output is lower-bounded by

$$a_j^{\mathrm{b,out}(1)} \geq d_{\mathrm{free}}^{\mathrm{o}} \alpha^{\mathrm{i}} j + (d_{\mathrm{free}}^{\mathrm{o}} - 1)\beta^{\mathrm{b,i}} + \beta_1^{\mathrm{b,i}}, \tag{34}$$

where α^{i} *and* $\beta^{\mathrm{b,i}}$ *are the minima of the two active distance-slopes and the coefficients* $\beta_i^{\mathrm{b,i}}$ *for the constituent inner encoders, respectively, and* $d_{\mathrm{free}}^{\mathrm{o}}$ *is the free distance of the outer encoder.*

Proof. See Appendix.

If, instead of the free distance we write the minimum free distance among the constituent encoders in the corresponding warp, we obtain lower bounds on the active input- and output-distances of a twill.

4.2. Double-Warp Woven Convolutional Encoders

As we have seen in the previous sections, the mapping between information and code symbols plays an important role in unequal error protection. Furthermore, it becomes even more important in cascaded encoding schemes. Consider a modification of a woven convolutional encoder which we introduced in [15]. In this encoder we kept the outer encoders the same but changed the way how the information symbols from the buffer are fed to the constituent encoders in the inner warp. Now the sequence is subdivided into segments according to the number of inputs of the constituent inner encoders, b_l^{i}, and fed to them block-wise. After encoding by the inner encoders, the resulting code sequences are combined block-wise. Later on, a twill-like woven encoder with the inner warp as specified above will be referred to as a woven encoder with a double-warp.

If the outer warp of a double-warp woven encoder consists of L_{o} identical encoders with (semi-infinite) encoding matrices G^{o}, and the inner warp consists of L_{i} identical encoders with (semi-infinite) encoding matrices G^{i}, then such a double-warp has the following (semi-infinite) encoding matrix

$$G^{\mathrm{dw}} = (G^{\mathrm{o}} \otimes I_{L_{\mathrm{o}}}) \left(I_{L_{\mathrm{i}}} \otimes G^{\mathrm{i}}\right). \tag{35}$$

where $I_{L_{\mathrm{o}}}$, $I_{L_{\mathrm{i}}}$ are the $L_{\mathrm{o}} \times L_{\mathrm{o}}$ and $L_{\mathrm{i}} \times L_{\mathrm{i}}$ identity matrices, respectively. Using the property of the Kronecker product that when all matrix products are defined [13]

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \tag{36}$$

the (semi-infinite) encoding matrix for the double-warp can be written as

$$G^{\mathrm{dw}} = G^{\mathrm{o}} \otimes G^{\mathrm{i}}. \tag{37}$$

Thus, this scheme is a generalization of a product code.

If the blocking factors B^{ow} and B^{iw} are chosen as given in (22), the encoding matrix for a double-warp with identical constituent encoders in the warps can be written as

$$G^{\mathrm{dw}}(D) = \Big(G^{\mathrm{o}}(D) \otimes I_{L_{\mathrm{o}}}\Big)^{[B^{\mathrm{ow}}]} \Big(I_{L_{\mathrm{i}}} \otimes G^{\mathrm{i}}(D)\Big)^{[B^{\mathrm{iw}}]} \tag{38}$$

If different encoders are used inside the warps, then it transforms into

$$G^{\mathrm{dw}}(D) = \left(\sum_{l=1}^{L_{\mathrm{o}}} G_l^{\mathrm{o}}(D) \otimes \mathrm{diag}(\boldsymbol{e}_l)\right)^{[B^{\mathrm{ow}}]} \left(\sum_{l=1}^{L_{\mathrm{i}}} \mathrm{diag}(\boldsymbol{e}_l) \otimes G_l^{\mathrm{i}}(D)\right)^{[B^{\mathrm{iw}}]} \tag{39}$$

where $\mathrm{diag}(\boldsymbol{e}_l)$ is a diagonal matrix with the lth unit vector on the main diagonal. For simplicity, we will always consider a situation when all constituent inner encoders have the same b_l^{i}.

Example 6. Consider a double-warp woven encoder with an outer warp consisting of two encoders with identical encoding matrices

$$G^{\mathrm{o}}(D) = \begin{pmatrix} 1 & 1+D \end{pmatrix} \tag{40}$$

and an inner warp consisting of two encoders with identical encoding matrices

$$G^{\mathrm{i}}(D) = \begin{pmatrix} D & 1+D & 1+D \\ 1 & D & 1+D \end{pmatrix}. \tag{41}$$

It is easy to verify that the blocking factors are $B^{\mathrm{ow}} = 1$ and $B^{\mathrm{iw}} = 1$, that is, we do not need any blocking to match the warps in this scheme.

According to (38), the encoding matrix for the double-warp woven encoder is

$$\begin{aligned} G^{\mathrm{dw}}(D) &= \Big(G^{\mathrm{o}}(D) \otimes I_2\Big)\Big(I_2 \otimes G^{\mathrm{i}}(D)\Big) \\ &= \begin{pmatrix} 1 & 0 & 1+D & 0 \\ 0 & 1 & 0 & 1+D \end{pmatrix} \begin{pmatrix} D & 1+D & 1+D & 0 & 0 & 0 \\ 1 & D & 1+D & 0 & 0 & 0 \\ 0 & 0 & 0 & D & 1+D & 1+D \\ 0 & 0 & 0 & 1 & D & 1+D \end{pmatrix} \\ &= \begin{pmatrix} D & 1+D & 1+D & D+D^2 & 1+D^2 & 1+D^2 \\ 1 & D & 1+D & 1+D & D+D^2 & 1+D^2 \end{pmatrix} \end{aligned}$$

A simple calculation of the Kronecker product $G^{\mathrm{o}}(D) \otimes G^{\mathrm{i}}(D)$ confirms (37). Note that despite L_{o} and L_{i} are not relatively prime, $G^{\mathrm{dw}}(D)$ does not correspond to several parallelly connected independent encoders as in Example 5.

For this double-warp woven scheme we are also able to lower-bound the active burst input- and output-distances, that is, to give counterparts of Theorem 1 and Theorem 2. Since we did not change the outer warp there is no difference between the twill and the double-warp, the former theorem is valid for the double-warp scheme. For a woven encoder with inner warp one has to take into account that the information sequence for the inner warp is distributed by blocks of size b^{i}. Thus, to exploit the free distance of the outer encoder we need b^{i} times fewer encoders in the inner warp compared to (32). At every time instant the outer encoder outputs either none or at least $d_{\mathrm{free}}^{\mathrm{o}}$ nonzero symbols. In the latter case, these symbols are fed to at least $\lceil d_{\mathrm{free}}^{\mathrm{o}}/b^{\mathrm{i}} \rceil$ constituent encoders. Hence, we have the following theorem.

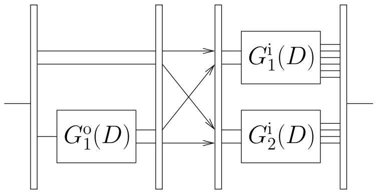

Figure 13. A simple woven convolutional encoder.

Theorem 3. *Consider the special case of a double warp woven convolutional encoder in which the outer warp consists of a single outer encoder with rate* $R^{\mathrm{o}} = b^{\mathrm{o}}/c^{\mathrm{o}}$ *encoding matrix* $G^{\mathrm{o}}(D)$ *and an inner warp consisting of* $L_{\mathrm{i},1}$ *encoders for rate* $R_1^{\mathrm{i}} = b_1^{\mathrm{i}}/c_1^{\mathrm{i}}$ *encoding matrices* $G_1^{\mathrm{i}}(D)$ *and* $L_{\mathrm{i},2}$ *encoders for rate* $R_2^{\mathrm{i}} = b_2^{\mathrm{i}}/c_2^{\mathrm{i}}$ *encoding matrices* $G_2^{\mathrm{i}}(D)$. *Assume that the total number of constituent inner encoders* $L_{\mathrm{i}} = L_{\mathrm{i},1} + L_{\mathrm{i},2}$ *satisfies*

$$L_{\mathrm{i}} \geq \left(j_{\mathrm{free}}^{\mathrm{s,o}} + 1\right) \frac{c^{\mathrm{o}}}{b^{\mathrm{i}}}. \tag{42}$$

If the following restriction is fulfilled

$$\min\left(\beta_1^{\mathrm{c,i}}, \beta_1^{\mathrm{rc,i}}, \beta_2^{\mathrm{c,i}}, \beta_2^{\mathrm{rc,i}}\right) \geq \alpha^{\mathrm{i}}, \tag{43}$$

then the active burst distance for the first output is lower-bounded by

$$a_j^{\mathrm{b,out}(1)} \geq \left\lceil \frac{d_{\mathrm{free}}^{\mathrm{o}}}{b^{\mathrm{i}}} \right\rceil \alpha^{\mathrm{i}} j + \left(\left\lceil \frac{d_{\mathrm{free}}^{\mathrm{o}}}{b^{\mathrm{i}}} \right\rceil - 1 \right) \beta^{\mathrm{b,i}} + \beta_1^{\mathrm{b,i}}, \tag{44}$$

where α^{i} *and* $\beta^{\mathrm{b,i}}$ *are the minima of the two active distance-slopes and the coefficients* $\beta_i^{\mathrm{b,i}}$ *for the constituent inner encoders, respectively, and* $d_{\mathrm{free}}^{\mathrm{o}}$ *is the free distance of the outer encoder.*

4.3. An Illuminative Example

As we have shown above, when the number of constituent encoders in the warps is sufficiently large, we can guarantee the achievement of unequal error protection for both information and code symbols. Nevertheless, even woven schemes with much smaller warps can provide different unequal error protection, meaning both different free input-distances (free output-distances) and different active input-distance-slopes (active output-distance-slopes).

Consider the simple woven construction from [1]. Its structure is shown in Figure 13. The constituent encoders are given by the following encoding matrices

$$G_1^{\mathrm{o}}(D) = \begin{pmatrix} 1 & 1+D \end{pmatrix} \tag{45}$$

$$G_1^{\mathrm{i}}(D) = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & D & D & 1 & 1 & 0 \end{pmatrix} \tag{46}$$

$$G_2^{\mathrm{i}}(D) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & D & 1 & 1 \end{pmatrix}. \tag{47}$$

Note that the encoding matrices for the constituent inner encoders for convenience of the realization were blocked [14] such that they have two inputs each; and they correspond to $G_1(D) = \begin{pmatrix} 1 & 1+D & D \end{pmatrix}$ and $G_2(D) = \begin{pmatrix} 1 & 1+D \end{pmatrix}$, respectively. The resulting encoding matrix for the woven encoder is

$$G_1^{\mathrm{wcc}}(D) = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & D & D & 1 & 1 & 0 & 0 & D+D^2 & 1+D & 1+D \end{pmatrix}. \tag{48}$$

	i	1	2	3
$G_1^{\mathrm{wcc}}(D)$	$d_{\mathrm{free}}^{\mathrm{in}(i)}$	4	3	9
	$\alpha^{\mathrm{in}(i)}$	18/5	3	3
$G_2^{\mathrm{wcc}}(D)$	$d_{\mathrm{free}}^{\mathrm{in}(i)}$	3	4	11
	$\alpha^{\mathrm{in}(i)}$	2	11/3	2
$G_3^{\mathrm{wcc}}(D)$	$d_{\mathrm{free}}^{\mathrm{in}(i)}$	4	4	8
	$\alpha^{\mathrm{in}(i)}$	7/2	2	7/2

Table 4. Free input-distances and active input-distance-slopes for our woven schemes.

	i	1	2	3	4	5	6	7	8	9	10
$G_1^{\mathrm{wcc}}(D)$	$d_{\mathrm{free}}^{\mathrm{out}(i)}$	4	4	9	9	4	4	3	3	9	3
	$\alpha^{\mathrm{out}(i)}$	18/5	13/4	3	3	3	18/5	3	3	3	3
$G_2^{\mathrm{wcc}}(D)$	$d_{\mathrm{free}}^{\mathrm{out}(i)}$	3	3	11	3	4	4	11	11	4	4
	$\alpha^{\mathrm{out}(i)}$	2	3	2	3	11/3	13/4	2	2	13/4	11/3
$G_3^{\mathrm{wcc}}(D)$	$d_{\mathrm{free}}^{\mathrm{out}(i)}$	4	4	8	8	4	4	4	4	4	8
	$\alpha^{\mathrm{out}(i)}$	7/2	10/3	7/2	7/2	10/3	7/2	2	2	2	7/2

Table 5. Free output-distances and active output-distance-slopes for our woven schemes.

It is minimal-basic [16] and can be realized in controller canonical form with only two memory elements. Thus, realizing the constituent encoders separately, we get a nonminimal realization of $G_1^{\mathrm{wcc}}(D)$ as it requires three memory elements (one for each of the three constituent encoders).

Computing the active burst input- and output-distances for this woven scheme yields the curves shown in Figures 14 and 15. The values for the corresponding free input- and output-distances as well as the active distance-slopes are given in the first rows of Tables 4 and 5.

For this woven scheme, we can observe the effect of the constituent encoders used in both warps. The encoding matrix $G_1^{\mathrm{o}}(D)$ has better error-correcting capability than the two 1×1 identity matrices (i.e., no coding at all) in the first two rows of the outer warp. This yields a much larger free input-distance for the third input. The slopes are nearly the same due to the effect of the simple interleaver between the warps. The encoding matrix $G_1^{\mathrm{i}}(D)$ in the inner warp has the $d_{\mathrm{free}} = 4$, which is larger than that of $G_2^{\mathrm{i}}(D)$, $d_{\mathrm{free}} = 3$. As a consequence, the minimum of the free output-distances for the first six outputs corresponding to $G_1^{\mathrm{i}}(D)$ is larger than that for the last four outputs. Nevertheless, the difference in active distance-slope for the outputs is not so large.

Woven schemes are very sensitive to the constituent encoders and to the order in which they are placed in the warps. To demonstrate this fact consider a woven scheme similar to what we had before but with exchanged order of $G_1^{\mathrm{i}}(D)$ and $G_2^{\mathrm{i}}(D)$. This new woven scheme has the following encoding matrix

$$G_2^{\mathrm{wcc}}(D) = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & D & 1 & 1 & 0 & D+D^2 & D+D^2 & 1+D & 1+D & 0 \end{pmatrix}. \tag{49}$$

The active burst distances for the inputs and outputs of $G_2^{\mathrm{wcc}}(D)$ are shown in Figures 16 and 17. The corresponding values of the free input- and output-distances as well as the active distance-slopes are written in Tables 4 and 5.

We see that even a simple change of the order of the two constituent encoders dramatically changes the active burst input- and output-distances.

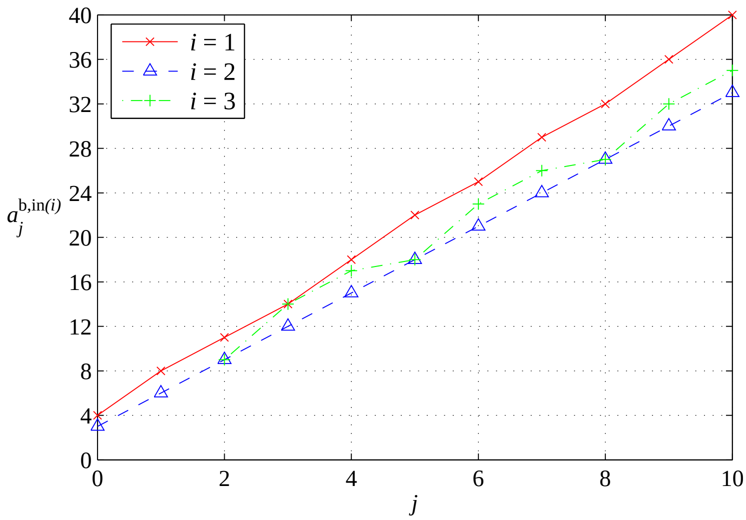

Figure 14. Active burst input-distances for the woven scheme with $G_1^{\text{wcc}}(D)$.

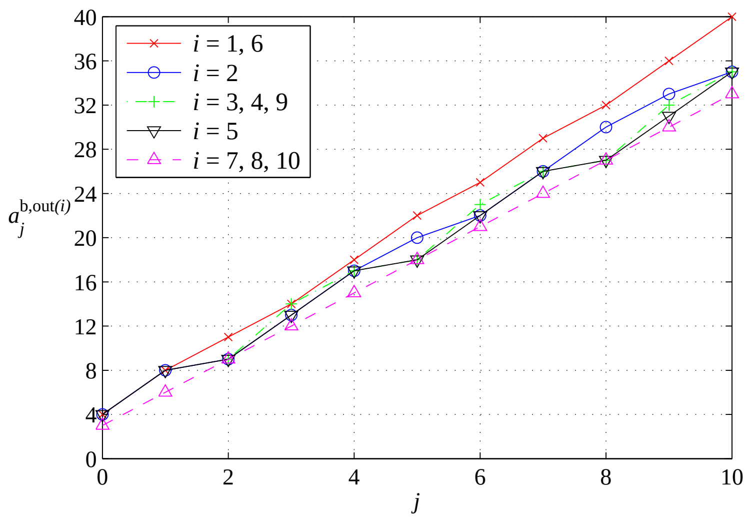

Figure 15. Active burst output-distances for the woven scheme with $G_1^{\text{wcc}}(D)$.

Consider now a woven scheme where the second constituent inner encoder corresponds to an encoding matrix $G_2(D) = \begin{pmatrix} \frac{1}{1+D} & 1 \end{pmatrix}$ blocked with factor $B = 2$, that is,

$$G_2^{\text{i}}(D) = \begin{pmatrix} \frac{1}{1+D} & 1 & \frac{1}{1+D} & 0 \\ \frac{D}{1+D} & 0 & \frac{1}{1+D} & 1 \end{pmatrix}. \tag{50}$$

Then the encoding matrix for the woven scheme is

$$G_3^{\text{wcc}}(D) = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{1+D} & 1 & \frac{1}{1+D} & 0 \\ 0 & D & D & 1 & 1 & 0 & D & 0 & 1 & 1+D \end{pmatrix}. \tag{51}$$

The active burst input- and output-distances are plotted in Figures 18 and 19. The corresponding values of the free input- and output-distances as well as the active input- and output-distance-slopes are written in Tables 4 and 5.

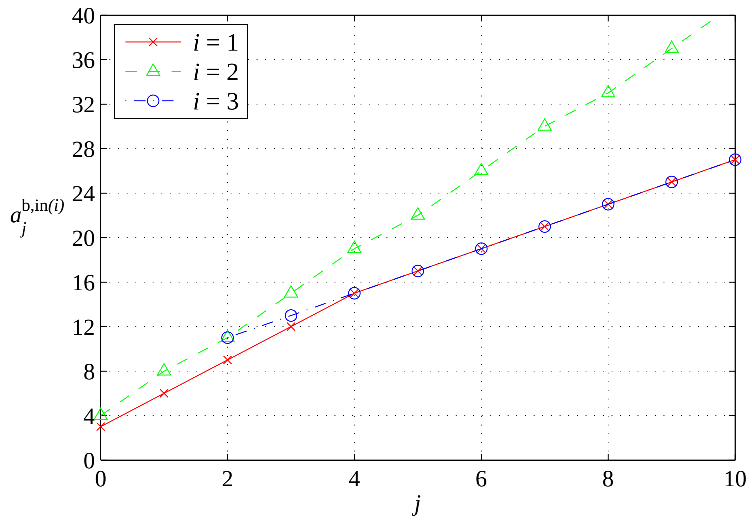

Figure 16. Active burst input-distances for the woven scheme with $G_2^{\mathrm{wcc}}(D)$.

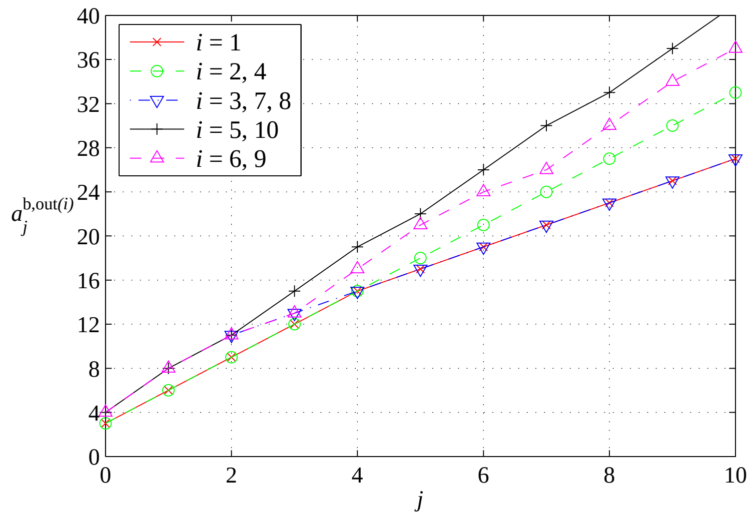

Figure 17. Active burst output-distances for the woven scheme with $G_2^{\mathrm{wcc}}(D)$.

Again we see that changing one of the constituent encoders to an equivalent systematic encoder changes all active distances for both inputs and outputs.

Through these examples we showed that it is possible to get different levels of protection both for the information symbols and for the code symbols via even simple woven schemes. For illustrative purposes the constituent encoders are chosen to be small. This causes only slight differences in the active distance-slopes between the different levels of the unequal error protection. In general, it is more difficult to obtain noticeable differences in the active distance-slopes than in the free input- and output-distances.

5. DECODING ISSUES

Modern communication systems tend to use longer codes. The coding schemes like turbo codes [17], woven convolutional codes, or low-density parity-check codes [18] are practically unfeasible for maximum-likelihood (ML) decoding and some suboptimum iterative decoding procedures are used instead. Suboptimum decoding adds one more degree for optimization of a system design as the error-correcting performance of different codes is not equally degraded.

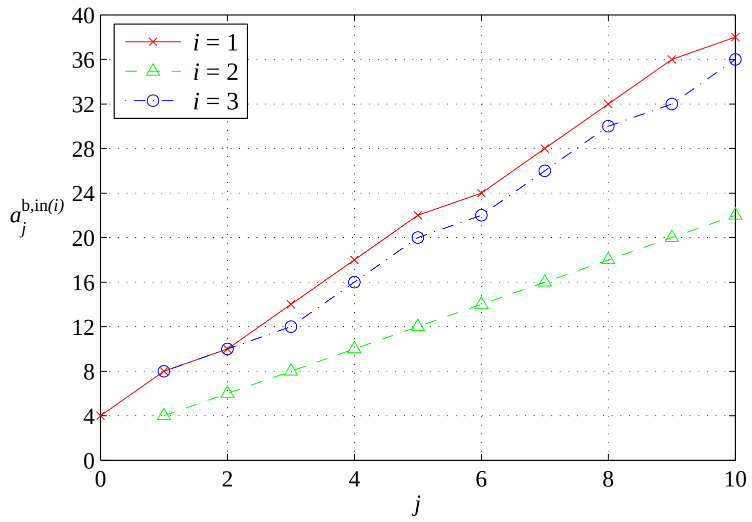

Figure 18. Active burst input-distances for the woven scheme with $G_3^{\mathrm{wcc}}(D)$.

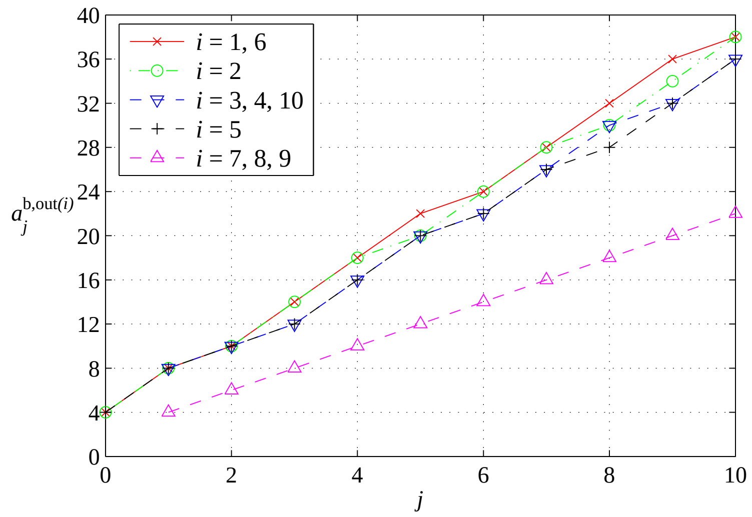

Figure 19. Active burst output-distances for the woven scheme with $G_3^{\mathrm{wcc}}(D)$.

In Figure 20 we present the results of the simulations of a woven encoder given in [19]. The woven convolutional encoder with outer warp consisting of one constituent encoder with encoding matrix

$$G_1^{\mathrm{o}}(D) = \begin{pmatrix} 1 & 1+D & 1+D^2 & 1+D+D^2 \end{pmatrix}$$

and as many as 24 constituent encoders with encoding matrices

$$G_2^{\mathrm{o}}(D) = \begin{pmatrix} 1+D & 1+D & 1 \\ 0 & D & 1+D \end{pmatrix}$$

and a single inner encoder with encoding matrix

$$G^{\mathrm{i}}(D) = \begin{pmatrix} 1 & \frac{1+D+D^2}{1+D^2} & \frac{1}{1+D^2} & 0 \\ 0 & \frac{D^2}{1+D^2} & \frac{1+D+D^2}{1+D^2} & 0 \\ 0 & \frac{D}{1+D^2} & \frac{D^2}{1+D^2} & 1 \end{pmatrix}.$$

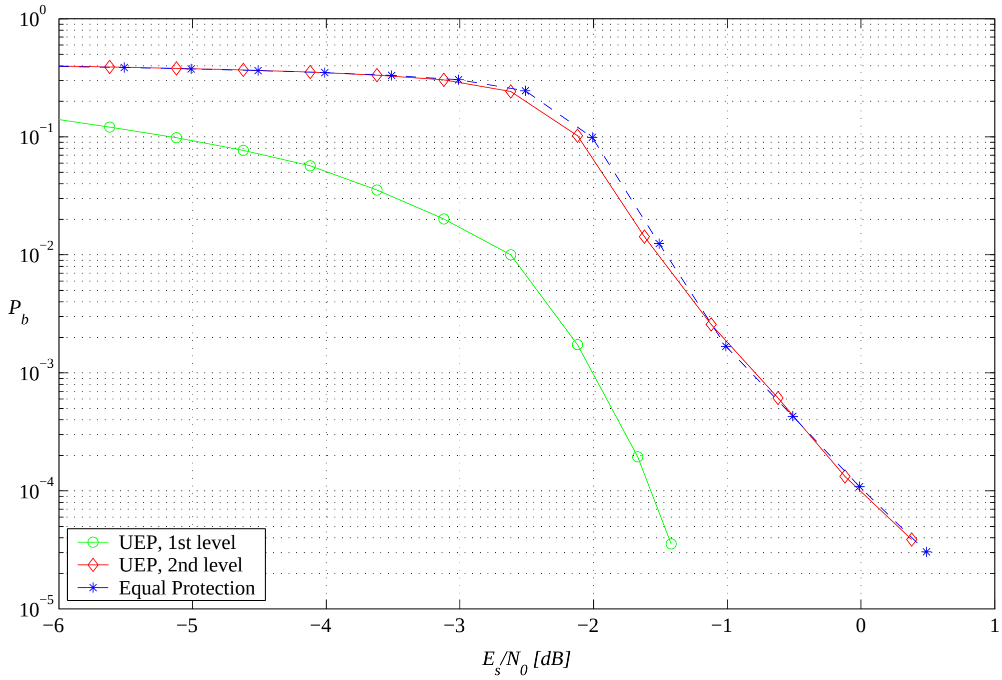

Figure 20. Bit-error rate curves for the simulation results given in [19].

For comparison, also a woven convolutional encoder with an outer warp consisting of 25 identical constituent encoders with encoding matrices $G_2^{\mathrm{o}}(D)$ and a single inner encoder with encoding matrix $G^{\mathrm{i}}(D)$ was simulated. A sliding window decoder [12] with 10 iterations was used.

Figure 20 clearly shows the presence of two different levels of information protection. According to the system parameters, for a relatively small fraction of the information symbols (approximately 1.5%) the system achieves $P_b = 10^{-4}$ at $E_s/N_0 = -1.6$ dB while for the rest of the information symbols we need $E_s/N_0 = 0$ dB. When lower P_b is required, the gap between them becomes even larger than 1.6 dB because the better protected fraction corresponds to a larger free input-distance, and, as a consequence, has steeper asymptotic slope. Note that in this case UEP is actually obtained at no cost since the curve for the normally protected symbols follows the reference curve for the system with equal error protection for the information symbols.

Other results of simulations of unequal error protection for information symbols in woven encoding schemes are presented in [20]. Despite that iterative APP decoding was used for the simulations, the resulting BER curves confirm the presence of unequal error protection.

APPENDIX

PROOFS

Proof of Theorem 1

The proof is based on and is similar to the proof of Theorem 8 in [11]. When the constituent encoders in the outer warp are considered as the inputs of the overall encoder, a weight of a detour for the first input can be lower-bounded by the minimum weight in a set of paths that start and terminate in the all-zero state and contain at least one segment with a nonzero input symbol for the first constituent encoder in the outer warp. If the number of the constituent encoders in the warp is large enough and satisfies (29), then the output weight of the inner encoder will pick up at least weight $d^{\mathrm{i}}_{\mathrm{free}}$ before the next code symbol from the same constituent encoder is fed to the inner encoder.

To lower-bound the active burst distance for the first input we consider several cases for the structure of the paths.

Case I: Suppose that the path causing the lower estimate of $a_j^{\mathrm{b,in}(1)}$ corresponds to a burst in the first constituent encoder in the warp over the whole length. Then if the number of constituent encoders in the warp satisfies (29), a burst generated by the first encoder is spread such that every nonzero symbol coming to the inner encoder causes a weight of at least $d^{\mathrm{i}}_{\mathrm{free}}$. Thus, we have

$$a_j^{\mathrm{b,in}(1)} \geq a_{j,1}^{\mathrm{b,o}} d^{\mathrm{i}}_{\mathrm{free}} \geq \left(\alpha_1^{\mathrm{o}} j + \beta_1^{\mathrm{b,o}}\right) d^{\mathrm{i}}_{\mathrm{free}} \geq d^{\mathrm{i}}_{\mathrm{free}} \alpha^{\mathrm{o}} j + d^{\mathrm{i}}_{\mathrm{free}} \beta_1^{\mathrm{b,o}} \tag{A.1}$$

where α^{o} denotes the minimum between α_1^{o} and α_2^{o}.

Case II: Suppose that the path causing the lower estimate of $a_j^{\mathrm{b,in}(1)}$ starts with a burst in the first constituent outer encoder but terminates with a burst in another constituent encoder with encoding matrix $G_1^{\mathrm{o}}(D)$. If (29) is satisfied, then every nonzero symbol from either one of the two constituent encoders causes a weight of at least $d^{\mathrm{i}}_{\mathrm{free}}$ in the output sequence of the woven encoder. During the time interval when these both constituent encoders have bursts, their nonzero symbols might appear at the input of the inner encoder too close to each other, and, thus, only the nonzero symbols from one of them cause a weight of $d^{\mathrm{i}}_{\mathrm{free}}$ each.

We can treat this case in two different ways. One can think that the first constituent encoder has a burst from time zero to $k+1 \leq j$, and when it remerges to the all-zero path, another constituent encoder has a burst until time instant $j+1$. Then the weight of the output sequence of the inner encoder is lower-bounded by

$$\begin{aligned} a_j^{\mathrm{b,in}(1)} &\geq \left(a_{k,1}^{\mathrm{b,o}} + a_{j-k-1,1}^{\mathrm{rc,o}} - 1\right) d^{\mathrm{i}}_{\mathrm{free}} \\ &\geq \left(\alpha_1^{\mathrm{o}} k + \beta_1^{\mathrm{b,o}} + \alpha_1^{\mathrm{o}}(j-k-1) + \beta_1^{\mathrm{rc,o}} - 1\right) d^{\mathrm{i}}_{\mathrm{free}} \\ &\geq \left(\alpha^{\mathrm{o}}(j-1) + \beta_1^{\mathrm{b,o}} + \beta_1^{\mathrm{rc,o}} - 1\right) d^{\mathrm{i}}_{\mathrm{free}}. \end{aligned} \tag{A.2}$$

If $\beta_1^{\mathrm{rc,o}} \geq \alpha^{\mathrm{o}} + 1$, then

$$a_j^{\mathrm{b,in}(1)} \geq d^{\mathrm{i}}_{\mathrm{free}} \alpha_1^{\mathrm{o}} j + d^{\mathrm{i}}_{\mathrm{free}} \beta_1^{\mathrm{b,o}}. \tag{A.3}$$

Alternatively, assume that the second part starts at time instant $n+1 \leq j$ when the first encoder still does not remerge with the all-zero path. Then we have

$$\begin{aligned} a_j^{\mathrm{b,in}(1)} &\geq \left(a_{n,1}^{\mathrm{c,o}} + a_{j-n-1,1}^{\mathrm{b,o}} - 1\right) d^{\mathrm{i}}_{\mathrm{free}} \\ &\geq \left(\alpha_1^{\mathrm{o}}(j-1) + \beta_1^{\mathrm{c,o}} + \beta_1^{\mathrm{b,o}} - 1\right) d^{\mathrm{i}}_{\mathrm{free}} \\ &\geq \left(\alpha^{\mathrm{o}}(j-1) + \beta_1^{\mathrm{c,o}} + \beta_1^{\mathrm{b,o}} - 1\right) d^{\mathrm{i}}_{\mathrm{free}}. \end{aligned} \tag{A.4}$$

If $\beta_1^{\mathrm{c,o}} \geq \alpha^{\mathrm{o}} + 1$, then $a_j^{\mathrm{b,in}(1)}$ is lower-bounded by

$$a_j^{\mathrm{b,in}(1)} \geq d^{\mathrm{i}}_{\mathrm{free}} \alpha^{\mathrm{o}} j + d^{\mathrm{i}}_{\mathrm{free}} \beta_1^{\mathrm{b,o}}. \tag{A.5}$$

Since these two treatments correspond to the same case, only one of the restrictions on the beta coefficients needs to be fulfilled and we get

$$\max\left(\beta_1^{\mathrm{c,o}}, \beta_1^{\mathrm{rc,o}}\right) \geq \alpha_1^{\mathrm{o}} + 1. \tag{A.6}$$

Case III: Suppose that the path causing the lower estimate of $a_j^{\mathrm{b,in}(1)}$ corresponds starts with a burst in the first constituent outer encoder but terminates with a burst in another constituent encoder with encoding matrix $G_2^{\mathrm{o}}(D)$. As before, suppose that (29) is satisfied and that the first constituent encoder has a burst from time zero to $k+1 \leq j$, and when it remerges to the all-zero path, another constituent encoder has a burst until time instant $j+1$. Then the weight of the output sequence of the inner encoder is lower-bounded by

$$\begin{aligned} a_j^{\mathrm{b,in}(1)} &\geq \left(a_{k,1}^{\mathrm{b,o}} + a_{j-k-1,2}^{\mathrm{rc,o}} - 1\right) d_{\mathrm{free}}^{\mathrm{i}} \\ &\geq \left(\alpha_1^{\mathrm{o}} k + \beta_1^{\mathrm{b,o}} + \alpha_2^{\mathrm{o}}(j-k-1) + \beta_2^{\mathrm{rc,o}} - 1\right) d_{\mathrm{free}}^{\mathrm{i}} \\ &\geq \left(\alpha^{\mathrm{o}}(j-1) + \beta_1^{\mathrm{b,o}} + \beta_2^{\mathrm{rc,o}} - 1\right) d_{\mathrm{free}}^{\mathrm{i}} \\ &\geq d_{\mathrm{free}}^{\mathrm{i}} \alpha^{\mathrm{o}} j + d_{\mathrm{free}}^{\mathrm{i}} \beta_1^{\mathrm{b,o}}, \end{aligned} \tag{A.7}$$

where the last inequality is valid if $\beta_2^{\mathrm{rc,o}} \geq \alpha^{\mathrm{o}} + 1$.

Assuming that the second part starts at time instant $n+1 \leq j$ when the first encoder still does not remerge with the all-zero path, we obtain

$$\begin{aligned} a_j^{\mathrm{b,in}(1)} &\geq \left(a_{n,1}^{\mathrm{c,o}} + a_{j-n-1,2}^{\mathrm{b,o}} - 1\right) d_{\mathrm{free}}^{\mathrm{i}} \\ &\geq \left(\alpha_1^{\mathrm{o}} n + \beta_1^{\mathrm{c,o}} + \alpha_2^{\mathrm{o}}(j-n-1) + \beta_2^{\mathrm{b,o}} - 1\right) d_{\mathrm{free}}^{\mathrm{i}} \\ &\geq \left(\alpha^{\mathrm{o}} j + \beta_1^{\mathrm{b,o}} + \beta_1^{\mathrm{c,o}} + \beta_2^{\mathrm{b,o}} - \beta_1^{\mathrm{b,o}} - \alpha^{\mathrm{o}} - 1\right) d_{\mathrm{free}}^{\mathrm{i}} \\ &\geq d_{\mathrm{free}}^{\mathrm{i}} \alpha^{\mathrm{o}} j + d_{\mathrm{free}}^{\mathrm{i}} \beta_1^{\mathrm{b,o}}, \end{aligned} \tag{A.8}$$

with the last inequality valid if $\beta_1^{\mathrm{c,o}} + \beta_2^{\mathrm{b,o}} - \beta_1^{\mathrm{b,o}} \geq \alpha^{\mathrm{o}} + 1$. Since these two restrictions on the beta coefficients correspond to the same case, it is sufficient if one of them is satisfied, that is,

$$\max\left(\beta_2^{\mathrm{rc,o}}, \beta_1^{\mathrm{c,o}} + \beta_2^{\mathrm{b,o}} - \beta_1^{\mathrm{b,o}}\right) \geq \alpha^{\mathrm{o}} + 1. \tag{A.9}$$

Case IV: Consider now the reversed case compared to Case II, that is, when an overall burst starts with a burst in one of the $L_{\mathrm{o},1} - 1$ constituent outer encoders with encoding matrix $G_1^{\mathrm{o}}(D)$, but terminates with the first constituent encoder. Since both encoders have coinciding encoding matrices $G_1^{\mathrm{o}}(D)$, we immediately have the same chains of inequalities (A.2) and (A.4) as well as the restriction (A.6) on them.

Case V: As a counterpart for Case III, consider a burst in the woven encoder starts with a burst in a constituent outer encoder with encoding matrix $G_2^{\mathrm{o}}(D)$, but terminates with the first constituent encoder. Analogously to the previous cases,

$$\begin{aligned} a_j^{\mathrm{b,in}(1)} &\geq \left(a_{k,2}^{\mathrm{b,o}} + a_{j-k-1,1}^{\mathrm{rc,o}} - 1\right) d_{\mathrm{free}}^{\mathrm{i}} \\ &\geq \left(\alpha_2^{\mathrm{o}} k + \beta_2^{\mathrm{b,o}} + \alpha_1^{\mathrm{o}}(j-k-1) + \beta_1^{\mathrm{rc,o}} - 1\right) d_{\mathrm{free}}^{\mathrm{i}} \\ &\geq \left(\alpha^{\mathrm{o}} j + \beta_1^{\mathrm{b,o}} + \beta_2^{\mathrm{b,o}} + \beta_1^{\mathrm{rc,o}} - \beta_1^{\mathrm{b,o}} - \alpha^{\mathrm{o}} - 1\right) d_{\mathrm{free}}^{\mathrm{i}} \\ &\geq d_{\mathrm{free}}^{\mathrm{i}} \alpha^{\mathrm{o}} j + d_{\mathrm{free}}^{\mathrm{i}} \beta_1^{\mathrm{b,o}}, \end{aligned} \tag{A.10}$$

with the last inequality valid if $\beta_2^{\mathrm{b,o}} + \beta_1^{\mathrm{rc,o}} - \beta_1^{\mathrm{b,o}} \geq \alpha^{\mathrm{o}} + 1$.

Alternatively, we have

$$\begin{aligned} a_j^{\mathrm{b,in}(1)} &\ge \left(a_{n,2}^{\mathrm{c,o}} + a_{j-n-1,1}^{\mathrm{b,o}} - 1\right) d_{\mathrm{free}}^{\mathrm{i}} \\ &\ge \left(\alpha_2^{\mathrm{o}} n + \beta_2^{\mathrm{c,o}} + \alpha_1^{\mathrm{o}}(j-n-1) + \beta_1^{\mathrm{b,o}} - 1\right) d_{\mathrm{free}}^{\mathrm{i}} \\ &\ge \left(\alpha^{\mathrm{o}} j + \beta_1^{\mathrm{b,o}} + \beta_2^{\mathrm{c,o}} - \alpha^{\mathrm{o}} - 1\right) d_{\mathrm{free}}^{\mathrm{i}} \\ &\ge d_{\mathrm{free}}^{\mathrm{i}} \alpha^{\mathrm{o}} j + d_{\mathrm{free}}^{\mathrm{i}} \beta_1^{\mathrm{b,o}}, \end{aligned} \tag{A.11}$$

with the last inequality valid if $\beta_2^{\mathrm{c,o}} \ge \alpha^{\mathrm{o}} + 1$. The last two restrictions on the beta coefficients can be jointly written as

$$\max\left(\beta_2^{\mathrm{b,o}} + \beta_1^{\mathrm{rc,o}} - \beta_1^{\mathrm{b,o}}, \beta_2^{\mathrm{c,o}}\right) \ge \alpha^{\mathrm{o}} + 1. \tag{A.12}$$

If we consider a path consisting of more than two segments corresponding to different constituent encoders, then we will get the same inequalities with increased number of the added coefficients β. All the restrictions that appeared in different cases have to be simultaneously satisfied. Then, a bit strengthened, they can be combined as

$$\min\left(\beta_1^{\mathrm{c,o}}, \beta_1^{\mathrm{rc,o}}, \beta_2^{\mathrm{c,o}}, \beta_2^{\mathrm{rc,o}}\right) \ge \alpha^{\mathrm{o}} + 1. \tag{A.13}$$

Hence, the statement of the theorem is proved.

Proof of Theorem 2

The proof is based on the proof of Theorem 11 in [11]. If the inner warp is large enough and satisfies (32), then at every time instant the outer encoder has at least $d_{\mathrm{free}}^{\mathrm{o}}$ nonzero symbols among L_{i} input sequences for the constituent inner encoders, keeping active every one of these $d_{\mathrm{free}}^{\mathrm{o}}$ encoders.

We start with the first constituent inner encoder. To lower-bound the contribution of the first constituent encoder, similarly to the proof of Theorem 1, we consider several cases.

Case I: Suppose that the first constituent inner encoder is involved in the overall burst over whole its length $j+1$. Then its contribution to the overall weight is

$$a_{j,1}^{\mathrm{b,i}} \ge \alpha_1^{\mathrm{i}} j + \beta_1^{\mathrm{b,i}} \ge \alpha^{\mathrm{i}} j + \beta_1^{\mathrm{b,i}}, \tag{A.14}$$

where α^{i} denotes the minimum between α_1^{i} and α_2^{i}.

Case II: Consider an overall burst that starts with a burst in the first constituent encoder but terminates with another constituent encoder with encoding matrix $G_1^{\mathrm{i}}(D)$.

This case can be treated in two different ways. Suppose that that the first constituent inner encoder has a burst from time zero to $k+1 \le j$, and when it remerges to the all-zero path, another constituent encoder with the encoding matrix $G_1^{\mathrm{i}}(D)$ has a burst until time instant $j+1$. Then that their joint contribution to the output weight is

$$\begin{aligned} a_{k,1}^{\mathrm{b,i}} + a_{j-k-1,1}^{\mathrm{rc,i}} &\ge \alpha_1^{\mathrm{i}}(j-1) + \beta_1^{\mathrm{b,i}} + \beta_1^{\mathrm{rc,i}} \\ &\ge \alpha^{\mathrm{i}}(j-1) + \beta_1^{\mathrm{b,i}} + \beta_1^{\mathrm{rc,i}} \\ &\ge \alpha_1^{\mathrm{i}} j + \beta_1^{\mathrm{b,i}}, \end{aligned} \tag{A.15}$$

with the last inequality valid if $\beta_1^{\mathrm{rc,i}} \ge \alpha^{\mathrm{i}}$.

Alternatively, we can suppose that the first constituent inner encoder has a burst from time zero to at least $n+1 \le j$, when another constituent encoder with encoding matrix $G_1^{\mathrm{i}}(D)$ diverges from the all-zero path and has a burst until time instant $j+1$. They jointly contribute

$$\begin{aligned} a^{\mathrm{c,i}}_{n,1} + a^{\mathrm{b,i}}_{j-n-1,1} &\geq \alpha^{\mathrm{i}}_1(j-1) + \beta^{\mathrm{c,i}}_1 + \beta^{\mathrm{b,i}}_1 \\ &\geq \alpha^{\mathrm{i}}(j-1) + \beta^{\mathrm{c,i}}_1 + \beta^{\mathrm{b,i}}_1 \\ &\geq \alpha^{\mathrm{i}}_1 j + \beta^{\mathrm{b,i}}_1, \end{aligned} \tag{A.16}$$

with the last inequality valid if $\beta^{\mathrm{c,i}}_1 \geq \alpha^{\mathrm{i}}$. Since both these treatments correspond to the same case, only one of the restrictions on the beta coefficients need to be fulfilled, that is,

$$\max\left(\beta^{\mathrm{c,i}}_1, \beta^{\mathrm{rc,i}}_1\right) \geq \alpha^{\mathrm{i}}. \tag{A.17}$$

Case III: Suppose that an overall burst starts with a burst in the first constituent encoder but terminates with another constituent encoder with encoding matrix $G^{\mathrm{i}}_2(D)$. Then, repeating calculations from the previous case (up to an appropriate change of the indices), we obtain

$$\begin{aligned} a^{\mathrm{b,i}}_{k,1} + a^{\mathrm{rc,i}}_{j-k-1,2} &\geq \alpha^{\mathrm{i}}_1 k + \beta^{\mathrm{b,i}}_1 + \alpha^{\mathrm{i}}_2(j-k-1) + \beta^{\mathrm{rc,i}}_2 \\ &\geq \alpha^{\mathrm{i}}(j-1) + \beta^{\mathrm{b,i}}_1 + \beta^{\mathrm{rc,i}}_2 \\ &\geq \alpha^{\mathrm{i}}_1 j + \beta^{\mathrm{b,i}}_1, \end{aligned} \tag{A.18}$$

with the last inequality valid if $\beta^{\mathrm{rc,i}}_2 \geq \alpha^{\mathrm{i}}$.

Alternatively, the joint contribution of the two constituent inner encoders can be lower-bounded as

$$\begin{aligned} a^{\mathrm{c,i}}_{n,1} + a^{\mathrm{b,i}}_{j-n-1,2} &\geq \alpha^{\mathrm{i}}_1 n + \beta^{\mathrm{c,i}}_1 + \alpha^{\mathrm{i}}_2(j-n-1) + \beta^{\mathrm{b,i}}_2 \\ &\geq \alpha^{\mathrm{i}} j + \beta^{\mathrm{b,i}}_1 + \beta^{\mathrm{c,i}}_1 + \beta^{\mathrm{b,i}}_2 - \beta^{\mathrm{b,i}}_1 - \alpha^{\mathrm{i}} \\ &\geq \alpha^{\mathrm{i}}_1 j + \beta^{\mathrm{b,i}}_1, \end{aligned} \tag{A.19}$$

with the last inequality valid if $\beta^{\mathrm{c,i}}_1 + \beta^{\mathrm{b,i}}_2 - \beta^{\mathrm{b,i}}_1 \geq \alpha^{\mathrm{i}}$.

Keeping in mind that the last two restrictions on the beta coefficients correspond to the same case, they can be combined as

$$\max\left(\beta^{\mathrm{rc,i}}_2, \beta^{\mathrm{c,i}}_1 + \beta^{\mathrm{b,i}}_2 - \beta^{\mathrm{b,i}}_1\right) \geq \alpha^{\mathrm{i}}. \tag{A.20}$$

Case IV: Suppose now a case reversed compared to Case II, that is, when an overall burst starts with a burst in one of the $L_{\mathrm{i},1}-1$ constituent inner encoders with encoding matrix $G^{\mathrm{i}}_1(D)$, but terminates with the first constituent encoder. Since both encoders have coinciding encoding matrices $G^{\mathrm{i}}_1(D)$, we immediately have the same chains of inequalities (A.15) and (A.16) as well as the restriction (A.17) on them.

Case V: Changing the order of the segments corresponding to different constituent encoders in Case III, we obtain the following inequalities

$$\begin{aligned} a^{\mathrm{b,i}}_{k,2} + a^{\mathrm{rc,i}}_{j-k-1,1} &\geq \alpha^{\mathrm{i}}_2 k + \beta^{\mathrm{b,i}}_2 + \alpha^{\mathrm{i}}_1(j-k-1) + \beta^{\mathrm{rc,i}}_1 \\ &\geq \alpha^{\mathrm{i}} j + \beta^{\mathrm{b,i}}_1 + \beta^{\mathrm{b,i}}_2 + \beta^{\mathrm{rc,i}}_1 - \beta^{\mathrm{b,i}}_1 - \alpha^{\mathrm{i}} \\ &\geq \alpha^{\mathrm{i}}_1 j + \beta^{\mathrm{b,i}}_1, \end{aligned} \tag{A.21}$$

with the last inequality valid if $\beta^{\mathrm{rc,i}}_1 + \beta^{\mathrm{b,i}}_2 - \beta^{\mathrm{b,i}}_1 \geq \alpha^{\mathrm{i}}$, or, alternatively,

$$\begin{aligned} a^{\mathrm{c,i}}_{n,2} + a^{\mathrm{b,i}}_{j-n-1,1} &\geq \alpha^{\mathrm{i}}_2 n + \beta^{\mathrm{c,i}}_2 + \alpha^{\mathrm{i}}_1(j-n-1) + \beta^{\mathrm{b,i}}_1 \\ &\geq \alpha^{\mathrm{i}}(j-1) + \beta^{\mathrm{c,i}}_2 + \beta^{\mathrm{b,i}}_1 \\ &\geq \alpha^{\mathrm{i}}_1 j + \beta^{\mathrm{b,i}}_1, \end{aligned} \tag{A.22}$$

with the last inequality valid if $\beta_2^{\mathrm{c,i}} \geq \alpha^{\mathrm{i}}$. The requirements for the beta coefficients can be joined as

$$\max\left(\beta_1^{\mathrm{rc,i}} + \beta_2^{\mathrm{b,i}} - \beta_1^{\mathrm{b,i}}, \beta_2^{\mathrm{c,i}}\right) \geq \alpha^{\mathrm{i}}. \tag{A.23}$$

All other cases of more consisting of more then two segments corresponding to different consistent encoders yield larger estimates of the contribution of the first constituent encoder due to increased number of the added beta coefficients. Thus, we do not have to consider them. Summarizing all the cases above, we can strengthen a bit the requirements on beta coefficients and combine them as

$$\min\left(\beta_1^{\mathrm{c,i}}, \beta_1^{\mathrm{rc,i}}, \beta_2^{\mathrm{c,i}}, \beta_2^{\mathrm{rc,i}}\right) \geq \alpha^{\mathrm{i}}. \tag{A.24}$$

Since at any time instant at least $d^{\mathrm{o}}_{\mathrm{free}}$ inner encoders are active, for the remaining $d^{\mathrm{o}}_{\mathrm{free}} - 1$ constituent inner encoders we can lower-bound every of them by

$$\alpha^{\mathrm{i}} j + \beta^{\mathrm{b,i}}, \tag{A.25}$$

where $\beta^{\mathrm{b,i}} = \min\left\{\beta_1^{\mathrm{b,i}}, \beta_2^{\mathrm{b,i}}\right\}$. Thus, we conclude that if both (32) and (A.24) are satisfied, then the active burst distance for the first output of the woven encoder is lower-bounded by

$$a_j^{\mathrm{b,out(1)}} \geq d^{\mathrm{o}}_{\mathrm{free}} \alpha^{\mathrm{i}} j + (d^{\mathrm{o}}_{\mathrm{free}} - 1)\beta^{\mathrm{b,i}} + \beta_1^{\mathrm{b,i}}. \tag{A.26}$$

This finishes the proof.

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